A Fresh Look at Peg Solitaire

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Peg solitaire is a one-person game that is over 300 years old; most people are familiar with the puzzle on the "standard 33-hole board" in FIGURE 1. When I first saw this game, what struck me was the unusual shape of the board. How was this strange cross-shaped board discovered and what is so special about it? While the history of the game is too fragmented to answer the question of the origin of this board, this paper will demonstrate that the special shape of the standard board can be derived from first principles. This board arises as a consequence of two very natural requirements: that of symmetry, and the ability to play from a board position with one peg missing to a single peg at the same location. We'll show that in a certain well-defined sense, the shape of this board is unique.

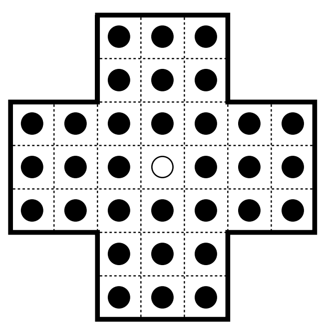

Figure 1 The standard 33-hole board

We refer to a board location as a **hole** because a physical board contains a hole or depression, in which the peg (or marble) sits. In all the diagrams, a hole with a peg is denoted by the symbol ●, while an empty hole is denoted by the symbol ○. The game begins with a peg in every hole except one, shown as the central hole in FIGURE 1. The player then jumps one peg over another into an empty hole on the board, removing the peg jumped over. No diagonal jumps are allowed, and the goal is to finish with one peg.

On the standard board, it is possible to start from the position in FIGURE 1 and finish with one peg in the center. Such a peg solitaire problem is called a **complement problem** because the starting and ending board positions are complements of one another (where every peg is replaced by a hole and vice versa). Note that all complement problems in this paper (by definition) start with one peg missing and finish with one peg.

In general, a board can be any region of holes on a square lattice. However the most aesthetically pleasing boards are those with some kind of symmetry.

Board symmetry

The highest degree of symmetry for a board (on a square lattice) is square symmetry. A **square-symmetric** board is unchanged by a reflection about either axis or either 45°

diagonal. Square-symmetric boards come in two varieties: **even** and **odd**, depending on whether their width is even or odd (or equivalently, the total number of holes T is even or odd). The standard 33-hole board is odd square-symmetric, and all such boards have a unique central hole. Even square-symmetric boards have a block of 4 central holes.

The pegs on odd boards can be divided into four categories: those that can reach the central hole, those that can jump over it vertically, those that can jump over it horizontally, and those that can neither reach it nor jump over it. Each peg remains in the same category for the entire game. On an even board, any peg can reach one and only one of the four central holes, and this also gives four categories of pegs. However the four jump patterns for an even board are simply reflections of one another. Because of this, in a general sense peg solitaire on even boards is less complex than on odd boards, and we expect that odd boards will produce more interesting and challenging problems.

We will use Cartesian coordinates to identify holes in a square-symmetric board, always placing the geometrical center of the board at the origin. On an odd board, the central hole is (0, 0), and all holes have integer coordinates. On an even board the four central holes are $(\pm 1/2, \pm 1/2)$, and all holes have half-integer coordinates. When we say one board is **smaller** than another, we always mean that the board has fewer holes.

A board is called **gapless** if, for any two holes on the board in the same column (or row), all the intervening holes are also on the board. This is equivalent to specifying that any horizontal or vertical line intersects the board either in a single interval, or not at all. Geometrically, saying a board is gapless is stronger than connectivity, but weaker than convexity. For example the standard 33-hole board is gapless (but not convex). Boards with interior voids or missing pieces along an edge are not gapless. Note that any jump must occur entirely on the board, and therefore if there is an interior void no jump is permitted into or over this void. For this reason boards that are not gapless can be cumbersome to play on, and we will consider only gapless boards, until the last section.

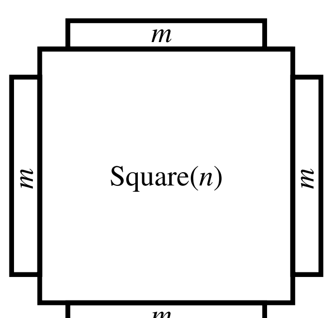

Figure 2 Augmenting a square-symmetric board.

The square board n holes on a side is certainly gapless and square-symmetric and will be called Square(n). What other gapless, square-symmetric boards are possible? Starting from Square(n), there is a geometrical technique to create a larger, square-symmetric board. We simply add a $1 \times m$ strip of holes symmetrically around all four sides as in FIGURE 2. To preserve square symmetry m must have the same parity as n, and we must have $m \leq n$ if the strips are not to overlap.

This process of adding strips of holes symmetrically to all four sides will be referred to as **augmenting** a board. Clearly we can repeat the process, adding another strip, and a whole (finite) sequence of strips of width m_i. In order that the final board be gapless, the integer sequence m_i must be *non-increasing*. When Square(n) is augmented by strips of width m_i, we'll denote the resulting board by Square(n) + (m_i). Let $\mathcal{B}$ be the set of all boards obtained by this construction.

$$\mathcal{B} = \{\text{Square}(n) + (m_i) \mid 0 < m_i \leq n \text{ non-increasing and } m_i \equiv n \pmod 2\}$$

PROPOSITION 1. *The set $\mathcal{B}$ contains* ***all*** *gapless, square-symmetric boards.*

Proof. By construction every board in $\mathcal{B}$ is gapless and square-symmetric. Is it possible that there is a gapless, square-symmetric board B that is not in $\mathcal{B}$? No, it isn't possible, because the gapless property ensures that the edge of the board must be formed from contiguous strips of holes, so we can remove them to obtain a smaller board that is still gapless and square-symmetric. We can continue this reduction inductively and it must terminate at a square board, so B $\in \mathcal{B}$. ■

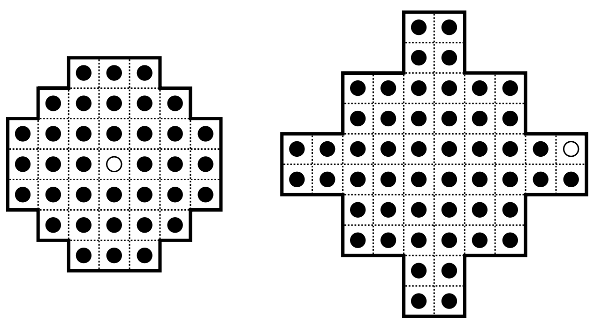

Figure 3 Sample elements of $\mathcal{B}$: (a) Square(5) + (3), known as the "French" board, (b) Square(6) + (2, 2).

FIGURE 3 shows two sample elements of $\mathcal{B}$. In this notation, the standard board of FIGURE 1 is Square(3) + (3, 3). Note that Square(n) + (m_i) has $T = n^2 + 4\sum m_i$ holes.

Null-class boards

Up until this section the rules of peg solitaire have not influenced the shape of the board, but we now determine properties that make for good peg solitaire boards. These stem from parity arguments along the diagonals [**1**], or alternatively the same theory can be derived from algebraic requirements [**2, 3**]. We use the former here because it is easier to understand the implications for square symmetry.

Consider two diagonal labelings of the holes of the board as shown in FIGURE 4 on square boards. Given a board position b, let $n_i(b)$ be the number of pegs in the holes marked i, and $t(b)$ be the total number of pegs on the board. A solitaire jump cannot change the parity of the differences $t - n_i$. This partitions the set of all possible board positions into **sixteen position classes** depending on the parity of the six integers $(t - n_i | i = 0, 1, \ldots, 5)$. Thus, all play is restricted to the position class of the starting position.

A **null-class** board is identified by the fact that b and the complement of b always lie in the same position class. In particular this must be true of the full and empty boards. We'll use the notation T and N_i for $t(b)$ and $n_i(b)$ when b is the full board.

T is the total number of holes in the board and N_i is the number of holes labeled i in FIGURE 4. The empty board lies in the position class where all six parities are even. Therefore a null-class board is one for which the six numbers $T - N_i$ are all even, or equivalently, all six N_i have the same parity (all odd or all even).

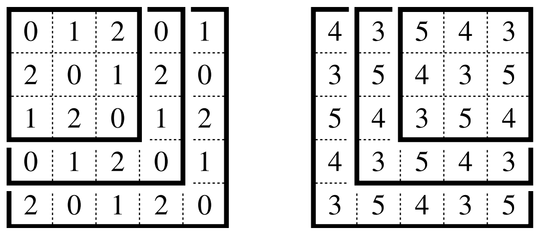

Figure 4 The labeling of holes on Square(n) where $n = 3$, 4 or 5.

For Square(3), we can see that all six $N_i = 3$, therefore this board is null-class. For $n = 4$ or 5, there is always an extra "0," or $N_0 = N_1 + 1$, and these boards are not null-class. In general, Square(n) is null-class if and only if n is a multiple of 3.

More interesting is the fact that *the process of augmenting a square board does not alter whether it is null class or not*. Why is this the case? If the augmentation process adds the hole (x_h, y_h), then it also adds the hole (y_h, x_h) reflected across the diagonal line $x = y$. The process never adds holes along the diagonal $x = y$, which ensures that $x_h \neq y_h$, so the holes are distinct. Because the parity labeling of FIGURE 4a is symmetric about the diagonal $x = y$, the two holes (x_h, y_h) and (y_h, x_h) are labeled the same, so *holes are always added in pairs with the same parity labels*. Therefore the parity of N_i does not change when the board is augmented. This completes a proof of the following proposition.

PROPOSITION 2. *Square(n) + (m_i) $\in \mathcal{B}$ is null-class if and only if n is a multiple of* 3.

Universal solvability

Why is null-class so important? Only on a null-class board can a board position and its complement be in the same position class. Therefore *a complement problem can only be solvable on a null-class board*. For this reason, null-class boards are the most interesting peg solitaire boards.

By Proposition 2, we know that the 37-hole "French" board of FIGURE 3a is not null-class and therefore no complement problem is solvable. In fact, the starting position for the central or $(0, 0)$ complement problem is in the position class of the empty board, and cannot be reduced to a single peg, anywhere. The impossibility of solving a central vacancy to one peg is shared by **all** elements of $\mathcal{B}$ for which n is not a multiple of 3.

Just because a board is null-class, however, does not imply that any complement problem is solvable. In general we must investigate the particular board more fully to answer this question. We will call a board **universally solvable** if the complement problem is solvable at every board location.

The goal of the remainder of this paper is to determine which elements of $\mathcal{B}$ are universally solvable. This would appear an ambitious goal, because the task is not easy even for the standard 33-hole board (which is universally solvable). Nonetheless, we shall see that significant progress can be made. Null-class is a necessary condition for complement problem solvability, so we now concentrate on boards for which n is a multiple of 3.

PROPOSITION 3. *The* $(0, 0)$ *complement problem is unsolvable on Square*$(3) + (1, 1, \ldots, 1)$ *or Square*$(3) + (3, 1, 1, \ldots, 1)$. *Here the sequence of consecutive 1's can have any length from zero to any positive integer.*

Proof. For boards of the first type, the $(0, 0)$ complement problem is clearly unsolvable, because there is no way to remove the peg at $(1, 1)$. For the boards of the second type, we use the resource count, or Pagoda Function shown in FIGURE 5. This is a real valued function of board position that (by construction) cannot increase during play. To calculate the value of this resource count for a particular board position, one sums the numbers where a peg is present. The reader should verify that no jump can increase the value of this resource count.

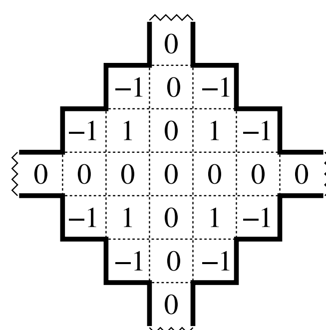

Figure 5 A resource count on Square(3) + (3, 1, 1, ..., 1).

For the central complement problem, this resource count begins at -4 and ends at 0; since solitaire jumps cannot increase the value of a resource count, it is impossible to reach the final state. In fact the same argument gives a much stronger result: no matter which peg is removed at the start, it is impossible to finish with fewer than 3 pegs. ∎

THEOREM 1. *The standard* 33-*hole board Square*$(3) + (3, 3)$ *is the smallest square-symmetric, gapless board that is universally solvable.*

Proof. This is immediate from Propositions 2 and 3, because the only null-class members of $\mathcal{B}$ that are smaller than the standard 33-hole board are those covered by Proposition 3. It is well known that the standard 33-hole board is universally solvable [**1, 2**]. ∎

We can also identify the next largest universally solvable element of $\mathcal{B}$, the 36-hole board Square(6). This board is less interesting than the standard board due to its simpler geometry and the fact that it is even square-symmetric. Many other universally solvable boards can also be created by augmenting this board, such as Square$(6) + (m_i)$, where $(m_i) = (2), (2, 2), (4), (4, 2)$ or (6). We can show this by finding solutions to all complement problems.

Experienced peg solitaire players know that on the standard 33-hole board, the most difficult complement problem to solve is the $(3, 0)$ complement (or symmetric equivalents). To obtain further intuition about larger boards, let us consider Wiegleb's board,

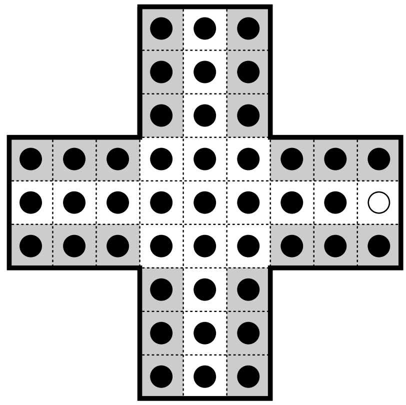

Figure 6 The board Square(3) + (3, 3, 3), "Wiegleb's Board." The significance of the shading will be explained in the proof of Theorem 3.

shown in FIGURE 6. This board was first introduced by J. Wiegleb in 1779 [**5**], but has since been relatively ignored.

Beasley [**1**, p. 200] states that all complement problems on Wiegleb's board are solvable *except* for the $(4, 0)$ complement problem, with starting position shown in FIGURE 6 (or symmetric equivalents). The difficulty of solving the $(4, 0)$ complement on Wiegleb's board is in fact a problem seen in all elements of $\mathcal{B}$: the most difficult complement problem to solve *begins from the center of the tip of the "arm."* Another example is the complement problem with starting position shown in FIGURE 3b, this problem is solvable but is the most difficult to solve on this board.

This suggests a useful generalization: we isolate the rightmost 3×3 section of Wiegleb's board (called "the needle" in the next section), and try to understand why the complement problem starting at the tip is difficult. The rest of the board (left of this 3×3 section) is not as important, and we can even allow it to be arbitrary. To solve the tip complement problem we must remove most of the pegs in the needle, but somehow build a trail of pegs to facilitate the final jumps back into the tip.

Boards with needles

Here we consider the general situation where a board of arbitrary shape has a $j \times m$ rectangular "needle" in the right half-plane $x > 0$, as in FIGURE 7b. The board in this

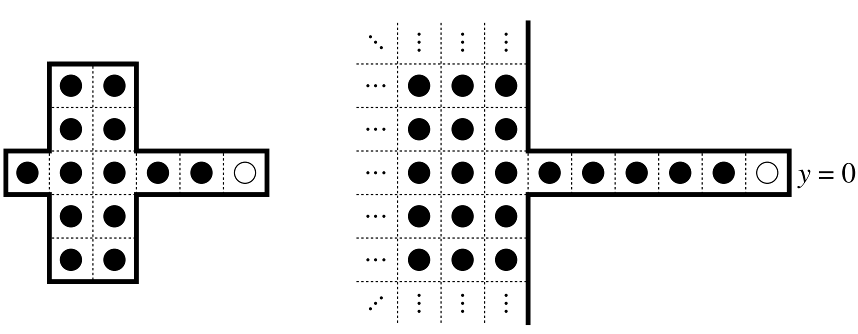

Figure 7 (a) A board containing a 1×3 needle. (b) A 1×6 needle attached to an arbitrary board.

section is not assumed to have any symmetry. Note that we have moved the coordinate origin to the base of the needle.

Let us first consider the needle of width $j = 1$. We want to answer the question: can we find a board containing a $1 \times m$ needle such that the complement problem at the tip of the needle is solvable? FIGURE 7a provides an example for the case $m = 3$. The reader should find the jumps for a solution to the tip complement problem; it will help to understand the problem and how to solve longer needles. The first two jumps are forced, after that you will find yourself doing a lot of rightward jumps to try to get a peg back to the end of the needle. The 1×4 and 1×5 needles are more difficult and require successively larger boards. Can we always solve longer needles by making the board larger and larger? No, as we will soon prove, the tip complement problem on a 1×6 needle is always unsolvable, no matter what board it is attached to. Although the right-half of the board is 1-dimensional, the left half is arbitrary, so this is not true 1D peg solitaire [**4**].

This problem is closely related to the "solitaire army" problem, a peg solitaire problem played on an infinite board [**1, 2**]. The solitaire army problem begins from a similar board position as FIGURE 7b, with pegs filling the entire left half-plane $x \leq 0$, and the goal is to jump a peg as far to the right as possible. The surprising result [**1, 2**] is that it is impossible to sent a scout (or peg) out 5 holes, no matter how many pegs are used. This result has been generalized to n-dimensions and diagonal jumps [**6, 7**], as well as other starting configurations [**8**].

Although similar to the solitaire army problem, our tip complement problem differs in several respects. Most significantly, there are pegs in the right half-plane at the start. More subtly, we cannot make any jump which is off the board, such as a rightward jump over (0, 1) in FIGURE 7b. Nonetheless, a similar technique is used to prove the following theorem.

THEOREM 2. *On any board with a $j \times m$ needle, for $j = 1$, 2 or 3 and $m > 5$, the tip or $(m, 0)$ complement problem is unsolvable.*

Proof. Consider the case $j = 1$ and the 1×6 needle of FIGURE 7b. We'll prove that the (6, 0) complement problem can't be solved (note that any longer needle can be considered a special case). To accomplish this, we use the resource count of FIGURE 8 (for the moment, ignore the values that are off the board). Let σ be the positive root of $x^2 + x - 1$, i.e. $\sigma = \frac{1}{2}(\sqrt{5} - 1) \approx .618$. σ is the reciprocal of the classical golden ratio. By construction $\sigma^2 + \sigma = 1$, and therefore

$$\sigma^i + \sigma^{i-1} = \sigma^{i-2} \qquad i \in \mathbb{Z} \tag{1}$$

$\ddots$	$\vdots$	$\vdots$	$\vdots$						
$\cdots$	σ^{10}	σ^9	σ^8						
$\cdots$	σ^9	σ^8	σ^7	σ^6	σ^5	σ^4	σ^3	σ^2	σ
$\cdots$	σ^8	σ^7	σ^6	σ^5	σ^4	σ^3	σ^2	σ	1
$\cdots$	σ^9	σ^8	σ^7	σ^6	σ^5	σ^4	σ^3	σ^2	σ
$\cdots$	σ^{10}	σ^9	σ^8						
$\iddots$	$\vdots$	$\vdots$	$\vdots$						

Figure 8 The resource count for the needle boards

It is this property that makes the pattern in FIGURE 8 a valid resource count, i.e., no jump can increase its total. In fact, rightward jumps lose nothing by (1), and the only jumps that reduce the total are

1. Leftward jumps, which lose an amount twice the hole jumped over
2. Vertical jumps away from $y = 0$, which lose an amount twice the hole jumped over
3. Vertical jumps over $y = 0$, which lose an amount equal to the hole jumped over

We can also express powers of σ by the formula

$$\sigma^i = (-1)^i[F_{i-1} - F_i\sigma] \qquad i \in \mathbb{Z} \tag{2}$$

Where F_i are the Fibonacci numbers, identified by $F_1 = F_2 = 1$, and $F_i = F_{i-2} + F_{i-1}$. Equation (2) can be proved by induction, and applies to all $i \in \mathbb{Z}$ if we extend the Fibonacci numbers by defining $F_0 = 0$, $F_{-i} = (-1)^{i+1}F_i$.

Now let us compute the total resource count in FIGURE 8 over the starting position in FIGURE 7b. First, we have the useful summing formula

$$\sum_{i=a}^{b} \sigma^i = \frac{\sigma^a - \sigma^{b+1}}{1-\sigma} = \sigma^{a-2} - \sigma^{b-1} \qquad a \le b \tag{3}$$

The sum of all the values in the column $x = 0$, by (3) and (1), is

$$\sum_{i=6}^{\infty} \sigma^i + \sum_{i=7}^{\infty} \sigma^i = \sigma^4 + \sigma^5 = \sigma^3$$

Therefore the initial value of the resource count for the $(6, 0)$ complement starting position is, using (3) and (2) is

$$\sum_{i=3}^{\infty} \sigma^i + \sum_{i=1}^{5} \sigma^i = \sigma + (\sigma^{-1} - \sigma^4) = 5\sigma - 1 \tag{4}$$

In reality the board is finite, and (4) provides an upper bound on the initial value of the resource count. If the initial value of the resource count (or an upper bound) minus the amount lost by any required jumps is less than the value of the final position, the problem is unsolvable. This computation will be called the **solvability criterion**: the problem is unsolvable if

$$\begin{bmatrix} \text{initial} \\ \text{resource} \\ \text{count} \end{bmatrix} - \begin{bmatrix} \text{amount lost} \\ \text{by required} \\ \text{jumps} \end{bmatrix} - \begin{bmatrix} \text{final} \\ \text{resource} \\ \text{count} \end{bmatrix} < 0$$

Note that after the first jump, there will be a peg at $(6, 0)$ and this hole must be cleared before the final jump. The only possibility is a leftward jump over $(5, 0)$, which loses 2σ in resource count. So the solvability criterion gives

$$[5\sigma - 1] - [2\sigma] - [1] = 3\sigma - 2 = -\sigma^4 < 0$$

Therefore the tip complement problem on the 1×6 needle is unsolvable.

For the case $j = 2$, we extend the 1-needle board of FIGURE 8 to include the holes at $(1 - 6, 1)$. The starting resource count value is given by (4), plus the amount added by the six additional holes. This amount is

$$\sum_{i=1}^{6} \sigma^i = \sigma^{-1} - \sigma^5 = 4 - 4\sigma \tag{5}$$

Combining (4) and (5) the starting value of the resource count is $3+\sigma$. After the first jump, we must clear the pegs at (6, 0) and (6, 1), which can only be accomplished by leftward jumps over (5, 0) and (5, 1), losing 2σ and $2\sigma^2$. It seems we must have additional leftward jumps to remove the pegs along $y = 1$, but how can we be sure? This is answered neatly by the **exit theorems**, first stated by Beasley [**1**, p. 117], or see [**2**, p. 829]. One exit theorem states that any **region** of the board with at least 3 holes that starts out full but finishes empty must have at least two exits. An **exit** is any jump that removes a peg from the region and ends outside it. The first jump that removes a peg from the region must be an exit, and so must the last one.

Consider the region $R_1 = (4-6, 1)$. This region starts out full and finishes empty, so must have two exits, and these can only be the leftward jumps over (4, 1) or (3, 1), which each lose at least $2\sigma^4$. Likewise the region $R_2 = (2-6, 1)$ must have two exits, and these cannot be the same exits as for R_1. For R_2 we require two leftward jumps over (2, 1) or (1, 1), which each lose at least $2\sigma^6$. The solvability criterion therefore gives:

$$[3+\sigma] - [2\sigma + 2\sigma^2 + 4\sigma^4 + 4\sigma^6] - [1] = 45\sigma - 28 = -(3\sigma^6 + \sigma^8) < 0$$

So the (6, 0) complement problem on the 2×6 needle is unsolvable.

The final case is $j = 3$; this adds another row of holes at $y = -1$. The initial resource count value, from (4) and (5), is $7 - 3\sigma$. The big change is that we now can clear (6, 0) with a vertical jump, let us suppose it is cleared by an upward jump. We then must have two leftward jumps over (5, 1), and as exits from R_1 and R_2 we can use two leftward jumps over (3, 1) and (1, 1) as before. In addition we require one leftward jump over $(4, -1)$, and for the regions $R_3 = (3-5, -1)$ and $R_4 = (1-5, -1)$ two exit jumps over $(2, -1)$ and $(0, -1)$. If we tally all this up, the solvability criterion yields

$$\begin{aligned}[7-3\sigma] - [1 + 4\sigma^2 + 4\sigma^4 + 4\sigma^6 + 2\sigma^3 + 4\sigma^5 + 4\sigma^7] - [1] &= 19 - 31\sigma \\ &= -(2\sigma^7 + \sigma^5) < 0\end{aligned}$$

In this case leftward jumps are not the only possible exits for the four regions. We can use two downward jumps over (4, 0) as exits for both R_1 and R_2, which lose $2\sigma^2$, and two upward jumps over (3, 0) as exits for both R_3 and R_4, which lose $2\sigma^3$. The solvability criterion then gives

$$[7-3\sigma] - [1 + 6\sigma^2 + 4\sigma^3] - [1] = 3 - 5\sigma = -\sigma^5 < 0$$

We can also try clearing (6, 0) with a leftward jump, but the solvability condition is again negative. The (6, 0) complement on the 3×6 needle cannot be solved. ■

Is $m > 5$ in Theorem 2 the best possible bound? FIGURE 9 shows a 56-hole board with a 1×5 needle where the tip complement problem is solvable (in fact this board is universally solvable). A 75-hole board with a 3×5 needle with solvable tip complement problem can be found in [**9**]. The 2×5 needle is the most difficult of the three—the smallest known board has 134 holes. Square-symmetric examples can be found in Square(15) + (1, 1, 1, 1, 1) and Square(15) + (3, 3, 3, 3, 3).

What about needles of width $m = 4$ and beyond? Notice that any hole in a 4-needle has some horizontal and vertical jump into it. This extra freedom should allow us to find universally solvable examples that are as long as we like. For example, the 4×6 rectangular board by itself is universally solvable [**1**, p. 184]. It is not difficult to show that the $4 \times m$ rectangular board is universally solvable for any $m \geq 6$ that is a multiple of 3 (using “packages and purges [**2**, p. 807]”).

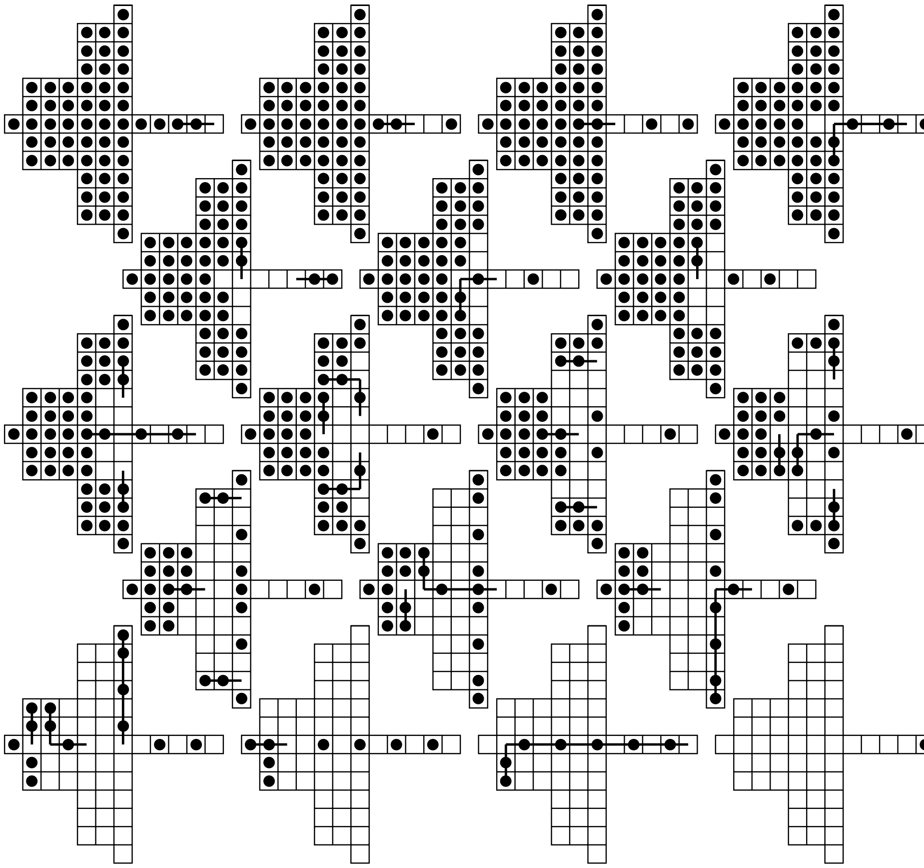

Figure 9 A board with a 1×5 needle with solvable tip complement problem.

Back to square symmetry

We can now use the results on needle boards to show that the standard board is even more remarkable.

THEOREM 3. *The standard* 33*-hole board is the only universally solvable board in* $\mathcal{B}$ *of the form Square*$(3) + (m_i)$.

Proof. Here m_i must have the form $(3, 3, \ldots, 3, 1, 1, \ldots, 1)$; let n_3 be the number of 3's in m_i and n_1 the number of 1's. To prove this theorem, it suffices to show that the $(n_3 + n_1 + 1, 0)$ complement problem at the tip of the "arm" is unsolvable, except for the case $n_3 = 2$; $n_1 = 0$. Many cases are proved unsolvable by Proposition 3 or Theorem 2. In fact, Theorem 2 can be further generalized to show that the tip complement problem on any board with $n_3 + n_1 > 5$ is unsolvable. The proof uses exactly the same techniques as Theorem 2, and we omit it. This leaves a total of nine special cases: $n_3 = 2$, $n_1 = 1, 2, 3$; $n_3 = 3$, $n_1 = 0, 1, 2$; $n_3 = 4$, $n_1 = 0, 1$; $n_3 = 5$, $n_1 = 0$.

The first three boards can be handled using the resource count of FIGURE 10 (note the Fibonacci numbers along the x-axis). For the $(4, 0)$ complement problem in FIGURE 10, this resource count begins at 45, and finishes at 21. But again the leftward jump to clear $(4, 0)$ loses 26, so the solvability criterion gives $[45] - [26] - [21] = -2 < 0$.

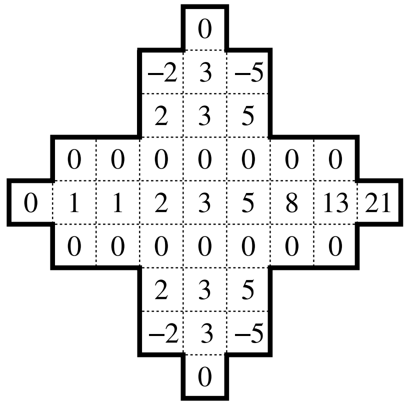

Figure 10 A resource count on Square(3) + (3, 3, 1)

This leaves six boards where the tip complement problem is difficult to prove unsolvable. For them, we use an integer programming (IP) model of the problem. We do not attempt to model the peg solitaire problem exactly (as in [**10**]), but allow any integer number of pegs in each hole, and a solitaire jump adds $(-1, -1, +1)$ to three consecutive holes. In this IP model, the order of the jumps is unimportant. For example, let's consider the (4, 0) complement problem on Wiegleb's board (FIGURE 6). On this board there are 108 geometrically possible jumps, and the number of each are our unknowns x_i. For each hole on the board, we have a linear equation which states that the starting number of pegs in this hole, minus the jumps that start from or jump over this hole, plus the jumps that end at this hole, equal the final number of pegs in this hole. This is a linear programming problem with 45 equations and 108 unknowns whose solution is restricted to non-negative integers, a standard problem for which computer solvers exist.

This IP model *is not* equivalent to the original peg solitaire problem, but it is solvable if the original problem is. Thus, if we can prove the IP model is unsolvable, it will prove the original problem unsolvable. Unfortunately, the (unmodified) IP model is solvable.

To complete the proof, we add to the IP model additional constraints that must be satisfied by the (4, 0) tip complement problem:

1. ≥ 2 rightward jumps into (4, 0) (the first and last jumps)
2. Exit requirements for each of the 8 shaded regions in FIGURE 6 (there are 5 possible exit jumps for each region)

When submitted to an integer programming solver (we recommend the free NEOS solver on the web [**11**]), the solver returns "integer infeasible." Similar computer proofs work for all 6 difficult boards. This is a rather subtle unsolvability, for if we take Wiegleb's board and remove the 3 holes at $x = -4$ the IP solver no longer reports that the (4, 0) complement is infeasible, and this 42-hole board can be shown to be universally solvable [**9**]. ∎

A final remarkable fact comes immediately from Theorem 3, since we know (or can determine) that Square(9) is universally solvable.

COROLLARY. *Among odd square-symmetric, gapless boards, the standard* 33*-hole board is the only board with less than* 81 *holes that is universally solvable.*

Boards with gaps

Here we consider the role of the gapless assumption in the above analysis. If we require only that a board be square-symmetric and null-class, what strange and interesting boards may result? First, it is easy to see that any board that is square-symmetric is either null-class, or can be made null-class by removing or adding the central hole(s) (one hole for an odd board, all 4 for an even board). Because of this the number of null-class, square-symmetric boards is large. However the vast majority are uninteresting to play peg solitaire on, for the board may not be connected, or there may be a hole into which no jump is possible.

A computer search for the smallest universally solvable, square-symmetric boards came up with the two boards in FIGURE 11. The reader may enjoy finding solutions to all complement problems on these boards.

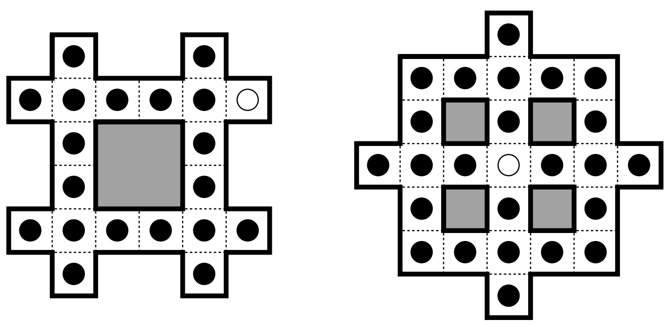

Figure 11 The smallest square-symmetric universally solvable boards (even and odd). Found by exhaustive computer search.

Conclusions

The concepts of null-class and symmetry provide a powerful combination for understanding peg solitaire boards. We have shown that the standard 33-hole board plays a special and unique role. It is the smallest gapless, square-symmetric board that is universally solvable. In fact it is even more special than this, because among gapless, odd square-symmetric boards, it is the only board with fewer than 81 holes that is universally solvable.

We should note that if we relax our symmetry requirements to rectangular symmetry, there are many universally solvable boards near the size of the standard board. For example, if we take the standard 33-hole board and remove the 6 holes at $y = \pm 3$, this 27-hole board is universally solvable. We can take Wiegleb's board and remove the 6 holes at $y = \pm 4$, this 39-hole board is also universally solvable, and the $(4, 0)$ complement problem has a *unique* solution, up to jump order and symmetry [**12**]. We can also play peg solitaire on a checkers board (allowing only diagonal jumps), this 32-hole board is universally solvable as well [**13**].

In this paper we have considered peg solitaire from a rather abstract perspective gained from years of exploration of the game, by hand and on a computer. We have given no actual solutions to problems, except for FIGURE 9. We hope the reader will be motivated to dust off a board (or find a computer version of the game) and try to solve the seven different complement problems on the standard board, and begin to explore problems on some of the other board shapes presented.

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REFERENCES

1. J. Beasley, *The Ins and Outs of Peg Solitaire*, Oxford Univ. Press, Oxford, New York, 1992.
2. E. Berlekamp, J. Conway, and R. Guy, *Winning Ways for your Mathematical Plays*, 2nd ed., A K Peters, Wellesley, MA, Vol. 4: 803–841, 2004.
3. A. Bialostocki, An application of elementary group theory to central solitaire, *Coll. Math. J.*, **29** (1998) 208–212.
4. C. Moore and D. Eppstein, 1-dimensional peg solitaire, and duotaire, in *More Games of No Chance* (R. Nowalowski, ed.), 2002, pp. 341–350.
5. J. Wiegleb, Anhang von dreyen Solitärspielen. Unterricht in der naturürlichen Magie (J. N. Martius), 1779, 413–416.
6. M. Aigner, Moving into the desert with Fibonacci, this MAGAZINE, **70** (1997) 11–21.
7. N. Eriksen, H. Eriksson, and K. Eriksson, Diagonal checker-jumping and Eulerian numbers for color signed permutations, *Electron. J. Combin.*, **7** (2000).
8. B. Csákány and R. Juhász, The solitaire army reinspected, this MAGAZINE, **73** (2000) 354–362.
9. J. Beasley, Games and Puzzles Journal #28, Special edition on peg solitaire, Sept. 2003, `http://www.gpj.connectfree.co.uk/gpjj.htm`.
10. C. Jefferson, A. Miguel, I. Miguel, and A. Tarim, Modelling and solving English peg solitaire, *Comp. and Op. Res.*, **33** (2006) 2935–2959.
11. J. Czyzyk, M. Mesnier, and J. Moré, The NEOS server, *IEEE J. Comp. Science and Eng.*, **5** (1998) 68–75, `http://neos.mcs.anl.gov/neos/`.
12. G. Bell and J. Beasley, New problems on old solitaire boards, *Board Game Studies*, **8** (2006) (to appear), `http://www.boardgamesstudies.org`.
13. B. Stewart, Solitaire on a checkerboard, *Amer. Math. Monthly*, **48** (1941) 228–233.