Estimating Radius of Gyration With a Bifilar Pendulum

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Introduction

The gyroscopic torques produced by propellers on model aircraft are relatively more important than those on larger aircraft primarily because of the higher rotational speeds they have. While carbon fiber propellers have provided a significant reduction in mass, the effect is still important.

When I began a study of this effect there had not, to my knowledge, been any systematic measurement of the rotary inertia of these small propellers. Because of this I performed a series of measurements and generated the kind of data required for estimating these effects on model aircraft. This note describes the device I constructed, the theory of its operation, and the method of reducing the test data.

Measurement of the moment of inertia of objects is required in many engineering applications. In aeronautics it has certainly been important and, in the years following World War I, measurement of the moment of inertia of the whole aircraft was required. The bifilar pendulum, described in this note, was used for this purpose.

I read of this device in an aeronautical engineering book published about 1930 and recognized that it built to the correct scale the device would be equally valuable for small components. In my case the propellers for “model” aircraft.

These propellers come in a large range of sizes, but for the ones used by internal combustion engines the diameters are primarily in the range of 10cm to 100cm with diameters in the range 15cm to 30cm being most important for the purpose I had in mind.

The last page of this note has a schematic of the pendulum which serves to show the dimensions on a simplified layout.

This note covers the analysis of the kind of bifilar pendulum I have constructed and used for this purpose.

Kinematics and Coordinates

The bifilar pendulum consists of two filaments of equal length L suspended from a level platform with a distance of D between filaments. Typically D is much smaller than L. A tray is attached to the end of the filaments and the object to be measured is place on the tray with the center of mass exactly mid way between the filaments. For the pendulum I used a medium weight thread was used. the primary restriction is hat the mass of the filament be negligible.

A small vertical wire that serves as a centering post is fixed to the ground below the tray. It is inserted in a hole the tray at the midpoint between the two suspending filaments. This hole is drilled slightly oversize compared to the wire to be inserted and is made exactly under the mass center of the object to be measured. This will prevent the pendulum from swinging back and fourth, constraining its motion to oscillation only. This vertical axis, in the plane of and parallel to the filaments at rest and at the same distance from each filament is called the 'axis' below.

A useful coordinate system is one that has an origin at the centering hole in the tray, the vertical dimension parallel to the axis, positive in the up direction, and one coordinate axis connecting the two filaments. In this system, with ‘up’ being the z axis and the line from the centering hole to the filament attachment point as the x , the filament attaches to the tray at the point $D/2$, 0, 0 and to the point of suspension $D/2$, 0, L. This system will be used in the non-linear analysis.

A small angle assumption is made in the analysis below. Although the nonlinear analysis is not hard to do, it does not make the simple relation among the important parameters easy to see and makes only small corrections to the answer. An analysis that does not make a small angle assumption is provided in an appendix.

When an object is placed in the tray and the tray rotated through an angle θ about the axis the ends of the tray move through a distance $D\theta/2$. If the angle θ is small, the vertical distance of the tray from the suspension point is reduced from L to $\sqrt{L^2 - (D\theta/2)^2}$

The potential energy of the combined tray and object whose combined mass is m is increased by an amount ΔE:

$$\Delta E = mg\left(L - \sqrt{L^2 - \left(\frac{D\theta}{2}\right)^2}\right)$$

$$\Delta E \simeq mg\frac{D^2\theta^2}{8L}$$

The combined tray and object with mass m has a radius of gyration r_g and moment of inertia $I = mr_g^2$. When oscillating in angle about the axis of the centering post the kinetic energy of the pendulum is :

$$E_k = mr_g^2\,\frac{\dot{\theta}^2}{2}$$

Conservation of energy dictates that :

$$mr_g^2\,\frac{\dot{\theta}^2}{2} + mg\frac{D^2\theta^2}{8L} = \text{const}$$

Differentiate the equation above to get :

$$r_g^2\,\ddot{\theta} + g\frac{D^2}{4L}\theta = 0$$

This is the equation of an undamped harmonic oscillator with a frequency of oscillation ω of

$$\omega = \frac{D}{2r_g}\sqrt{\frac{g}{L}}$$

Measurements of the period of oscillation, $P = 2\pi/\omega$, can be used to estimate the radius of gyration.

$$r_g = \frac{PD}{4\pi}\sqrt{\frac{g}{L}}$$

Note that this is the same relation as that for the linearized simple pendulum with small oscillations when all the mass is concentrated at the connection of the filament with the tray. In that case the radius of gyration is equivalent the distance $D/2$.

The full nonlinear equations are developed in the first appendix. This linear relation is satisfying in the sense of dimensional analysis in that the only measurements made are of time and length and the quantity estimated is a length. To estimate the moment of inertia the mass of the object must be determined independently

Practical Considerations

The mass and inertia of the tray alone must be determined. The tray can be weighed and then the period of oscillation of the tray alone is determined. The rotary inertia of the combined tray (subscript t) and propeller (subscript p) is simply $I = I_p + I_t$. If the radius of gyration and mass of the combined tray and propeller are known then:

$$r_p^2 m_p + r_t^3 m_t = r^2 (m_t + m_p)$$
$$r_p^2 = r^2 + (r^2 - r_t^2) \frac{m_t}{m_p}$$

In order to make the measurement of a period as accurate as possible one can place a mark at the point where the end of the tray has highest velocity and count (say) right-going passes of the tray. Measurements should be made over as many periods as is possible to make the period as accurate. In addition measurements should be repeated in a series of trials to assure they are consistent.

The pendulum should be suspended inside and away from any breeze. The pendulum I used had a filament of approximately 3 meters in length and a tray of about 35 cm length made from a piece of thin, hard-surfaced cardboard bonded to a matching size piece of foam to provide rigidity. Foam and cardboard were also used to make the attachment points on the pendulum and the mount on the ceiling.

Appendix: Nonlinear Analysis

Using the coordinate system described above and letting a 3-vector $\mathbf{L}$ with components of x, y, and z be written :

$$\mathbf{L} = \langle x,\, y,\, z, \rangle$$

Consider a filament with the tray at rest. The endpoints can be considered as vectors: at the suspension point $\mathbf{a} = \langle \frac{D}{2},\, 0,\, L, \rangle$ and at the tray $\mathbf{f} = \langle \frac{D}{2},\, 0,\, 0, \rangle$. The distance between these two points, the length of the filament is L.

When the tray is rotated through an angle θ the point at the tray becomes :

$$\mathbf{f} = \langle (D/2)\cos\theta,\, (D/2)\sin\theta,\, \delta z, \rangle$$

The distance between the two points remains the same

$$L^2 = \left[\frac{D}{2}(1-\cos\theta)\right]^2 + \left[\frac{D}{2}\cos\theta\right]^2 + (L-\delta z)^2$$

From which one has

$$\delta z = L - \tilde{L} = L - \sqrt{L^2 - 2(1-\cos\theta)\left(\frac{D}{2}\right)^2}$$

What is needed to substitute in the energy formula is the derivative of this with respect to time.

$$\frac{d\,\delta z}{d\,t} = \left(\frac{D}{2}\right)^2 \frac{\sin\theta}{\tilde{L}}$$

Where $\tilde{L}$ is very close (but slightly less than) L. If $D/L < 0.1$ Then for $\theta = \pi$ the error is 0.5%. The principal error, therefore is the difference between $\sin\theta$ and θ.

Compare this with the linearized version by dividing the result here by the linearized result:

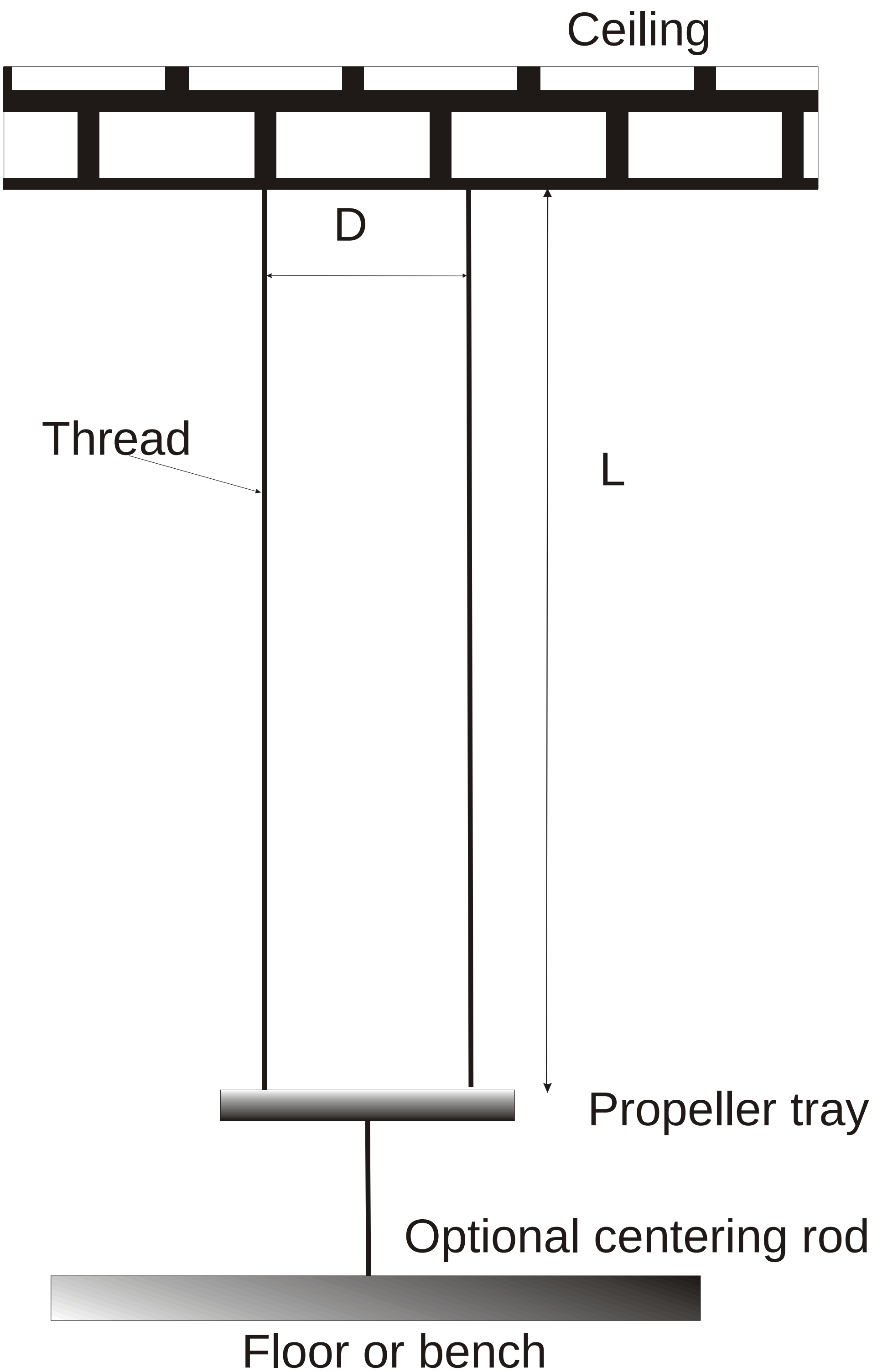
$$\frac{L}{\tilde{L}} \frac{\sin\theta}{\theta}$$

The differential equation is now

$$\ddot{\theta} + \frac{g}{\tilde{L}} \left(\frac{D}{2r_g} \right)^2 \sin\theta = 0$$

A very close approximation - assuming $\tilde{L} = L$ requires elliptic integrals for solution. This solution is a famous problem of the $19^t h$ century. For reference if the maximum angular deflection in 90 degrees the true period is 18% longer than that predicted by linear theory. For a 60 degree maximum it is a little over 7% and for a 30 degree swing it is about 1.4 %. Because the amplitude decays during the test the error will be less than the error on the first period.

Bifilar Pendulum



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