An inequality involving $2n$ numbers

Darij Grinberg

(version 22 August 2007)

1. The main inequality

In this note we are going to discuss two proofs and some applications of the following inequality:

Theorem 1.1. Let a_1, a_2, ..., a_n, b_1, b_2, ..., b_n be $2n$ reals. Assume that $\sum\limits_{1\leq i<j\leq n} a_i a_j \geq 0$ or[1] $\sum\limits_{1\leq i<j\leq n} b_i b_j \geq 0$. Then,

$$\left(\sum_{1\leq i\neq j\leq n} a_i b_j\right)^2 \geq 4 \sum_{1\leq i<j\leq n} a_i a_j \sum_{1\leq i<j\leq n} b_i b_j. \tag{1.1}$$

A remark about notation:

$$\sum_{1\leq i\neq j\leq n} \text{ is an abbreviation for } \sum_{1\leq i\leq n,\ 1\leq j\leq n,\ i\neq j}.$$

An important particular case of Theorem 1.1 is obtained when we set $n = 3$, $a_1 = a$, $a_2 = b$, $a_3 = c$, $b_1 = x$, $b_2 = y$, $b_3 = z$:

Theorem 1.2. Let a, b, c, x, y, z be six reals. Assume that $bc+ca+ab \geq 0$ or $yz + zx + xy \geq 0$. Then,

$$(ay + az + bz + bx + cx + cy)^2 \geq 4 (bc + ca + ab) (yz + zx + xy).$$

We are going to discuss in brief - and without proof - the equality case in Theorem 1.1. Before we can do this, we need to establish a notation:

The notation $(a_1, a_2, ..., a_n) \sim (b_1, b_2, ..., b_n)$ is going to mean that for every two numbers i and j from the set $\{1, 2, ..., n\}$, we have $a_i b_j = b_i a_j$. Note that if all numbers b_1, b_2, ..., b_n are nonzero, then $(a_1, a_2, ..., a_n) \sim (b_1, b_2, ..., b_n)$ is equivalent to $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = ... = \dfrac{a_n}{b_n}$.

Now, the question when equality holds in Theorem 1.1 can be answered:

Theorem 1.3. Let a_1, a_2, ..., a_n, b_1, b_2, ..., b_n be $2n$ reals. Assume that $\sum\limits_{1\leq i<j\leq n} a_i a_j \geq 0$ or $\sum\limits_{1\leq i<j\leq n} b_i b_j \geq 0$. Then, the inequality (1.1) becomes an equality if and only if (at least) one of the following three cases holds:

Case 1: We have $(a_1, a_2, ..., a_n) \sim (b_1, b_2, ..., b_n)$.

Case 2: We have $\sum\limits_{1\leq i\neq j\leq n} a_i b_j = 0$ and $\sum\limits_{1\leq i<j\leq n} a_i a_j = 0$.

Case 3: We have $\sum\limits_{1\leq i\neq j\leq n} a_i b_j = 0$ and $\sum\limits_{1\leq i<j\leq n} b_i b_j = 0$.

[1] Here and in the following, "or" means a logical "or". That is, when we say "$\mathcal{A}$ or $\mathcal{B}$", we mean "*at least one* of the two assertions $\mathcal{A}$ and $\mathcal{B}$ holds".

The proof of Theorem 1.3 is straightforward: Just follow our proofs of Theorem 1.1 and look out for possible equality cases.

Note that the 39th Yugoslav Federal Mathematical Competition 1998 featured a weaker version of Theorem 1.1 as problem 1 for the 3rd and 4th grades - weaker because it required the reals a_1, a_2, ..., a_n, b_1, b_2, ..., b_n to be nonnegative (while Theorem 1.1 only requires one of the two relations $\sum\limits_{1 \leq i < j \leq n} a_i a_j \geq 0$ and $\sum\limits_{1 \leq i < j \leq n} b_i b_j \geq 0$ to hold). The $n = 3$ case of this weaker version was discussed with a number of proofs in [1]. We are not going to focus on these weaker versions here, but rather show Theorem 1.1 in its general case.

2. Two proofs of Theorem 1.1

First proof of Theorem 1.1. The following proof of Theorem 1.1 is inspired by Sung-yoon Kim's post #5 in [1]. The crux is the following fact:

Theorem 2.1, the Aczel inequality. If a and b are two reals, and a_1, a_2, ..., a_n, b_1, b_2, ..., b_n are $2n$ reals such that $a^2 \geq \sum\limits_{k=1}^{n} a_k^2$, then

$$\left(ab - \sum_{k=1}^{n} a_k b_k\right)^2 \geq \left(a^2 - \sum_{k=1}^{n} a_k^2\right)\left(b^2 - \sum_{k=1}^{n} b_k^2\right). \qquad (2.1)$$

Proof of Theorem 2.1. Since $a^2 \geq \sum\limits_{k=1}^{n} a_k^2$, we have $a^2 - \sum\limits_{k=1}^{n} a_k^2 \geq 0$.

Now, if $b^2 - \sum\limits_{k=1}^{n} b_k^2 < 0$, then $\left(a^2 - \sum\limits_{k=1}^{n} a_k^2\right)\left(b^2 - \sum\limits_{k=1}^{n} b_k^2\right) \leq 0$ (since $a^2 - \sum\limits_{k=1}^{n} a_k^2 \geq 0$), so that (2.1) becomes trivial (since $\left(ab - \sum\limits_{k=1}^{n} a_k b_k\right)^2 \geq 0 \geq \left(a^2 - \sum\limits_{k=1}^{n} a_k^2\right)\left(b^2 - \sum\limits_{k=1}^{n} b_k^2\right)$). Thus, Theorem 2.1 is proven in the case when $b^2 - \sum\limits_{k=1}^{n} b_k^2 < 0$. It remains to prove Theorem 2.1 in the case when $b^2 - \sum\limits_{k=1}^{n} b_k^2 \geq 0$.

Consequently, we assume that $b^2 - \sum\limits_{k=1}^{n} b_k^2 \geq 0$ for the rest of this proof. Then, both numbers $a^2 - \sum\limits_{k=1}^{n} a_k^2$ and $b^2 - \sum\limits_{k=1}^{n} b_k^2$ are nonnegative, so that they have square roots. Now, the Cauchy-Schwarz inequality yields

$$\sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2 \geq \left(\sum_{k=1}^{n} a_k b_k\right)^2.$$

Taking the square root, we obtain

$$\sqrt{\sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2} \geq \left|\sum_{k=1}^{n} a_k b_k\right|. \qquad (2.2)$$

Hence,

$$|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{\left(\sum_{k=1}^n a_k^2 + \left(a^2 - \sum_{k=1}^n a_k^2\right)\right)\left(\sum_{k=1}^n b_k^2 + \left(b^2 - \sum_{k=1}^n b_k^2\right)\right)}$$

$$\geq \sqrt{\sum_{k=1}^n a_k^2 \cdot \sum_{k=1}^n b_k^2} + \sqrt{\left(a^2 - \sum_{k=1}^n a_k^2\right) \cdot \left(b^2 - \sum_{k=1}^n b_k^2\right)}$$

$\Big($by Cauchy-Schwarz in the form $\sqrt{(u+v)(u'+v')} \geq \sqrt{uu'} + \sqrt{vv'}$,

applied to $u = \sum_{k=1}^n a_k^2,\ v = a^2 - \sum_{k=1}^n a_k^2,\ u' = \sum_{k=1}^n b_k^2,\ v' = b^2 - \sum_{k=1}^n b_k^2$,

what is possible because these $u,\ v,\ u',\ v'$ are all nonnegative)

$$\geq \left|\sum_{k=1}^n a_k b_k\right| + \sqrt{\left(a^2 - \sum_{k=1}^n a_k^2\right) \cdot \left(b^2 - \sum_{k=1}^n b_k^2\right)} \qquad (\text{by } (2.2)),$$

so that

$$|ab| - \left|\sum_{k=1}^n a_k b_k\right| \geq \sqrt{\left(a^2 - \sum_{k=1}^n a_k^2\right) \cdot \left(b^2 - \sum_{k=1}^n b_k^2\right)}.$$

Since the right hand side of this inequality is ≥ 0 (because it is a square root), the left hand side must also be ≥ 0 (since it is greater or equal than the right hand side), and thus we can square this inequality. Upon squaring it, we obtain

$$\left(|ab| - \left|\sum_{k=1}^n a_k b_k\right|\right)^2 \geq \left(a^2 - \sum_{k=1}^n a_k^2\right) \cdot \left(b^2 - \sum_{k=1}^n b_k^2\right).$$

Since $|x-y| \geq ||x| - |y||$ for any two reals x and y, we have $\left|ab - \sum_{k=1}^n a_k b_k\right| \geq \left||ab| - \left|\sum_{k=1}^n a_k b_k\right|\right|$. Squaring this inequality, we obtain $\left(ab - \sum_{k=1}^n a_k b_k\right)^2 \geq \left(|ab| - \left|\sum_{k=1}^n a_k b_k\right|\right)^2$. Thus,

$$\left(ab - \sum_{k=1}^n a_k b_k\right)^2 \geq \left(|ab| - \left|\sum_{k=1}^n a_k b_k\right|\right)^2 \geq \left(a^2 - \sum_{k=1}^n a_k^2\right) \cdot \left(b^2 - \sum_{k=1}^n b_k^2\right),$$

and Theorem 2.1 is proven.

Now on to the proof of Theorem 1.1:

According to the condition of Theorem 1.1, we have $\sum_{1\leq i<j\leq n} a_i a_j \geq 0$ or $\sum_{1\leq i<j\leq n} b_i b_j \geq 0$. We can WLOG assume that $\sum_{1\leq i<j\leq n} a_i a_j \geq 0$ holds. Denote $a = \sum_{k=1}^n a_k$ and $b = \sum_{k=1}^n b_k$. Then,

$$a^2 = \left(\sum_{k=1}^n a_k\right)^2 = \sum_{k=1}^n a_k^2 + 2\underbrace{\sum_{1\leq i<j\leq n} a_i a_j}_{\geq 0} \geq \sum_{k=1}^n a_k^2.$$

Hence, we can apply Theorem 2.1 and obtain

$$\left(ab-\sum_{k=1}^{n}a_kb_k\right)^2 \geq \left(a^2-\sum_{k=1}^{n}a_k^2\right)\left(b^2-\sum_{k=1}^{n}b_k^2\right). \tag{2.3}$$

But

$$ab-\sum_{k=1}^{n}a_kb_k=\sum_{k=1}^{n}a_k\cdot\sum_{k=1}^{n}b_k-\sum_{k=1}^{n}a_kb_k=\sum_{1\leq i\leq n,\ 1\leq j\leq n}a_ib_j-\sum_{1\leq i=j\leq n}a_ib_j=\sum_{1\leq i\neq j\leq n}a_ib_j,$$

and also

$$a^2-\sum_{k=1}^{n}a_k^2=\left(\sum_{k=1}^{n}a_k\right)^2-\sum_{k=1}^{n}a_k^2=\left(\sum_{k=1}^{n}a_k^2+2\sum_{1\leq i<j\leq n}a_ia_j\right)-\sum_{k=1}^{n}a_k^2=2\sum_{1\leq i<j\leq n}a_ia_j,$$

and similarly

$$b^2-\sum_{k=1}^{n}b_k^2=2\sum_{1\leq i<j\leq n}b_ib_j.$$

Hence, (2.3) becomes

$$\left(\sum_{1\leq i\neq j\leq n}a_ib_j\right)^2\geq 2\sum_{1\leq i<j\leq n}a_ia_j\cdot 2\sum_{1\leq i<j\leq n}b_ib_j.$$

This is obviously equivalent to (1.1). Thus, (1.1) holds, so that Theorem 1.1 is proven.

Second proof of Theorem 1.1. We start with something trivial:

Lemma 2.2. If $u_1, u_2, ..., u_n$ are n reals such that $\sum\limits_{k=1}^{n}u_k=0$, then $\sum\limits_{1\leq i<j\leq n}u_iu_j\leq 0$.

Proof of Lemma 2.2. The condition $\sum\limits_{k=1}^{n}u_k=0$ yields

$$\begin{aligned}\sum_{k=1}^{n}u_k^2 &\geq 0 \qquad \text{(since a sum of squares is always } \geq 0\text{)}\\ &=0^2=\left(\sum_{k=1}^{n}u_k\right)^2=\sum_{k=1}^{n}u_k^2+2\sum_{1\leq i<j\leq n}u_iu_j,\end{aligned}$$

so that $0\geq 2\sum\limits_{1\leq i<j\leq n}u_iu_j$ and thus $\sum\limits_{1\leq i<j\leq n}u_iu_j\leq 0$. This proves Lemma 2.2.

Now to the proof of Theorem 1.1: According to the condition of Theorem 1.1, we have $\sum\limits_{1\leq i<j\leq n}a_ia_j\geq 0$ or $\sum\limits_{1\leq i<j\leq n}b_ib_j\geq 0$. We WLOG assume that $\sum\limits_{1\leq i<j\leq n}a_ia_j\geq 0$ holds.

If $\sum\limits_{k=1}^{n}a_k=0$, then Lemma 2.2 (applied to the reals $a_1, a_2, ..., a_n$ as $u_1, u_2, ..., u_n$) yields $\sum\limits_{1\leq i<j\leq n}a_ia_j\leq 0$, what, together with $\sum\limits_{1\leq i<j\leq n}a_ia_j\geq 0$, leads to $\sum\limits_{1\leq i<j\leq n}a_ia_j=0$,

so that the inequality (1.1) becomes trivial (because its left hand side, $\left(\sum\limits_{1\leq i\neq j\leq n} a_i b_j\right)^2$, is ≥ 0 since it is a square, and its right hand side, $4\sum\limits_{1\leq i<j\leq n} a_i a_j \sum\limits_{1\leq i<j\leq n} b_i b_j$, equals 0 because of $\sum\limits_{1\leq i<j\leq n} a_i a_j = 0$). Hence, Theorem 1.1 is proven in the case when $\sum\limits_{k=1}^{n} a_k = 0$.

Therefore, for the rest of our proof of Theorem 1.1, we will assume that $\sum\limits_{k=1}^{n} a_k \neq 0$. Then, we can define a real $t = \frac{\sum\limits_{k=1}^{n} b_k}{\sum\limits_{k=1}^{n} a_k}$, and set $c_i = b_i - ta_i$ for every $i \in \{1, 2, ..., n\}$.

Then,

$$\sum_{k=1}^{n} c_k = \sum_{k=1}^{n}$$

But

$$\left(\sum_{1\leq i\neq j\leq n} a_i c_j\right)^2 - 4\sum_{1\leq i<j\leq n} a_i a_j \sum_{1\leq i<j\leq n} c_i c_j$$

$$= \left(\sum_{1\leq i\neq j\leq n} a_i \left(b_j - t a_j\right)\right)^2 - 4\sum_{1\leq i<j\leq n} a_i a_j \sum_{1\leq i<j\leq n} \left(b_i - t a_i\right)\left(b_j - t a_j\right)$$

$$= \left(\sum_{1\leq i\neq j\leq n} \left(a_i b_j - t a_i a_j\right)\right)^2 - 4\sum_{1\leq i<j\leq n} a_i a_j \sum_{1\leq i<j\leq n} \left(b_i b_j + t^2 a_i a_j - t a_i b_j - t a_j b_i\right)$$

$$= \left(\sum_{1\leq i\neq j\leq n} a_i b_j - t\sum_{1\leq i\neq j\leq n} a_i a_j\right)^2$$

$$- 4\sum_{1\leq i<j\leq n} a_i a_j \left(\sum_{1\leq i<j\leq n} b_i b_j + t^2\sum_{1\leq i<j\leq n} a_i a_j - t\left(\sum_{1\leq i<j\leq n} a_i b_j + \sum_{1\leq i<j\leq n} a_j b_i\right)\right)$$

$$= \left(\sum_{1\leq i\neq j\leq n} a_i b_j - 2t\sum_{1\leq i<j\leq n} a_i a_j\right)^2 - 4\sum_{1\leq i<j\leq n} a_i a_j \left(\sum_{1\leq i<j\leq n} b_i b_j + t^2\sum_{1\leq i<j\leq n} a_i a_j - t\sum_{1\leq i\neq j\leq n} a_i b_j\right)$$

$$= \left(\sum_{1\leq i\neq j\leq n} a_i b_j\right)^2 - 4t\cdot\sum_{1\leq i\neq j\leq n} a_i b_j\cdot\sum_{1\leq i<j\leq n} a_i a_j + 4t^2\cdot\left(\sum_{1\leq i<j\leq n} a_i a_j\right)^2$$

$$- 4\sum_{1\leq i<j\leq n} a_i a_j\cdot\sum_{1\leq i<j\leq n} b_i b_j - 4t^2\cdot\left(\sum_{1\leq i<j\leq n} a_i a_j\right)^2 + 4t\cdot\sum_{1\leq i\neq j\leq n} a_i b_j\cdot\sum_{1\leq i<j\leq n} a_i a_j$$

$$= \left(\sum_{1\leq i\neq j\leq n} a_i b_j\right)^2 - 4\sum_{1\leq i<j\leq n} a_i a_j\cdot\sum_{1\leq i<j\leq n} b_i b_j.$$

Hence,

$$\left(\sum_{1\leq i\neq j\leq n} a_i b_j\right)^2 - 4\sum_{1\leq i<j\leq n} a_i a_j\cdot\sum_{1\leq i<j\leq n} b_i b_j \geq 0.$$

This immediately yields (1.1). Theorem 1.1 is therefore proved once again.

3. The first applications

The next paragraphs are devoted to various applications of Theorem 1.1. We start with a very easy one:

> **Theorem 3.1.** Let $r \geq 1$ be a real, and let a, b, c be three nonnegative reals satisfying $bc+ca+ab \geq 3$. Then, $a^r(b+c)+b^r(c+a)+c^r(a+b) \geq 6$.

Note that this theorem is a slightly extended version of [3], problem 5.2.14 and problem 8.2.21. The original source of this inequality is: Walther Janous and Vasile Cîrtoaje, CM, 5, 2003.

Proof of Theorem 3.1. Applying Theorem 1.2 for $x = a^r$, $y = b^r$, $z = c^r$ (obviously, $bc + ca + ab \geq 0$ holds because a, b, c are nonnegative), we get

$$\left(ab^r + ac^r + bc^r + ba^r + ca^r + cb^r\right)^2 \geq 4\left(bc + ca + ab\right)\left(b^r c^r + c^r a^r + a^r b^r\right).$$

This rewrites as

$$\left(a^r\left(b + c\right) + b^r\left(c + a\right) + c^r\left(a + b\right)\right)^2 \geq 4\left(bc + ca + ab\right)\left(\left(bc\right)^r + \left(ca\right)^r + \left(ab\right)^r\right).$$

After taking the square root, this becomes

$$a^r\left(b + c\right) + b^r\left(c + a\right) + c^r\left(a + b\right) \geq 2\sqrt{\left(bc + ca + ab\right)\left(\left(bc\right)^r + \left(ca\right)^r + \left(ab\right)^r\right)}.$$

Now, $bc+ca+ab \geq 3$, and since $r \geq 1$, the power mean inequality yields $\sqrt[r]{\dfrac{\left(bc\right)^r + \left(ca\right)^r + \left(ab\right)^r}{3}} \geq \dfrac{bc + ca + ab}{3} \geq \dfrac{3}{3} = 1$, so $\dfrac{\left(bc\right)^r + \left(ca\right)^r + \left(ab\right)^r}{3} \geq 1^r = 1$, so that $\left(bc\right)^r + \left(ca\right)^r + \left(ab\right)^r \geq 3$. Hence,

$$\begin{aligned} a^r\left(b + c\right) + b^r\left(c + a\right) + c^r\left(a + b\right) &\geq 2\sqrt{\left(bc + ca + ab\right)\left(\left(bc\right)^r + \left(ca\right)^r + \left(ab\right)^r\right)} \\ &\geq 2\sqrt{3 \cdot 3} = 6, \end{aligned}$$

and Theorem 3.1 is proven.

4. Walther Janous for n variables

Our next application is a generalization of a known inequality by Walther Janous. First we settle an auxiliary fact:

Theorem 4.1. Let x_1, x_2, ..., x_n be nonnegative real numbers such that $x_1 + x_2 + ... + x_n = 1$, and no $n - 1$ of these numbers are 0. Then,

$$\sum_{1 \leq i < j \leq n} \frac{x_i x_j}{\left(1 - x_i\right)\left(1 - x_j\right)} \geq \frac{n}{2\left(n - 1\right)}.$$

This Theorem 4.1 is problem 6.3.12 in [3], where it is proven using the Arithmetic Compensation Method, and is due to Gabriel Dospinescu (who is also known under the nickname Harazi).

Proof of Theorem 4.1. First,

$$\sum_{1 \leq i < j \leq n} \frac{x_i x_j}{\left(1 - x_i\right)\left(1 - x_j\right)} = \sum_{1 \leq i < j \leq n} \frac{\left(x_i x_j\right)^2}{x_i\left(1 - x_i\right) \cdot x_j\left(1 - x_j\right)}.$$

By the Cauchy-Schwarz inequality in the Engel form[2],

$$\sum_{1 \leq i < j \leq n} \frac{\left(x_i x_j\right)^2}{x_i\left(1 - x_i\right) \cdot x_j\left(1 - x_j\right)} \geq \frac{\left(\sum\limits_{1 \leq i < j \leq n} x_i x_j\right)^2}{\sum\limits_{1 \leq i < j \leq n} x_i\left(1 - x_i\right) \cdot x_j\left(1 - x_j\right)}.$$

[2]The *Cauchy-Schwarz inequality in the Engel form* is the inequality

$$\sum_{i=1}^n \frac{a_i^2}{b_i} \geq \frac{\left(a_1 + a_2 + ... + a_n\right)^2}{b_1 + b_2 + ... + b_n},$$

which holds for any n reals a_1, a_2, ..., a_n and any n positive reals b_1, b_2, ..., b_n.

Hence, in order to prove that

$$\sum_{1\leq i<j\leq n} \frac{x_i x_j}{(1-x_i)(1-x_j)} \geq \frac{n}{2(n-1)},$$

it remains to verify

$$\frac{\left(\sum\limits_{1\leq i<j\leq n} x_i x_j\right)^2}{\sum\limits_{1\leq i<j\leq n} x_i(1-x_i)\cdot x_j(1-x_j)} \geq \frac{n}{2(n-1)}. \tag{4.1}$$

But

$$\sum_{1\leq i<j\leq n} x_i x_j = \frac{1}{2}\cdot\left(\sum_{1\leq i<j\leq n} x_i x_j + \sum_{1\leq i<j\leq n} x_i x_j\right) = \frac{1}{2}\cdot\left(\sum_{1\leq i<j\leq n} x_i x_j + \sum_{1\leq j<i\leq n} x_i x_j\right)$$

$$= \frac{1}{2}\cdot\sum_{1\leq i\leq n,\ 1\leq j\leq n,\ i\neq j} x_i x_j = \frac{1}{2}\cdot\sum_{i=1}^{n} x_i \sum_{1\leq j\leq n,\ j\neq i} x_j$$

$$= \frac{1}{2}\cdot\sum_{i=1}^{n} x_i\left((x_1+x_2+...+x_n)-x_i\right) = \frac{1}{2}\cdot\sum_{i=1}^{n} x_i(1-x_i). \tag{4.2}$$

But for any n reals u_1, u_2, ..., u_n, we have

$$(u_1+u_2+...+u_n)^2 \geq \frac{2n}{n-1}\sum_{1\leq i<j\leq n} u_i u_j. \tag{4.3}$$

This can be verified as follows: We have

$$(u_1+u_2+...+u_n)^2 = \left(u_1^2+u_2^2+...+u_n^2\right)+2\sum_{1\leq i<j\leq n} u_i u_j \geq \frac{1}{n}(u_1+u_2+...+u_n)^2+2\sum_{1\leq i<j\leq n} u_i u_j$$

(because of the inequality $u_1^2+u_2^2+...+u_n^2 \geq \frac{1}{n}(u_1+u_2+...+u_n)^2$ that follows from QM-AM), so that

$$(u_1+u_2+...+u_n)^2 - \frac{1}{n}(u_1+u_2+...+u_n)^2 \geq 2\sum_{1\leq i<j\leq n} u_i u_j, \qquad \text{what becomes}$$

$$\frac{n-1}{n}\cdot(u_1+u_2+...+u_n)^2 \geq 2\sum_{1\leq i<j\leq n} u_i u_j, \qquad \text{what becomes}$$

$$(u_1+u_2+...+u_n)^2 \geq \frac{2n}{n-1}\sum_{1\leq i<j\leq n} u_i u_j,$$

and thus (4.3) is proven.

Now, according to (4.2), we have

$$\left(\sum_{1\leq i<j\leq n} x_i x_j\right)^2 = \left(\frac{1}{2}\cdot\sum_{i=1}^{n} x_i(1-x_i)\right)^2 = \frac{1}{4}\cdot\left(\sum_{i=1}^{n} x_i(1-x_i)\right)^2$$

$$\geq \frac{1}{4}\cdot\frac{2n}{n-1}\sum_{1\leq i<j\leq n} x_i(1-x_i)\cdot x_j(1-x_j) \qquad \text{(where we used (4.3) for } u_i = x_i(1-x_i)\text{)}$$

$$= \frac{n}{2(n-1)}\cdot\sum_{1\leq i<j\leq n} x_i(1-x_i)\cdot x_j(1-x_j),$$

so that

$$\frac{\left(\sum\limits_{1\leq i<j\leq n} x_i x_j\right)^2}{\sum\limits_{1\leq i<j\leq n} x_i\left(1-x_i\right)\cdot x_j\left(1-x_j\right)} \geq \frac{n}{2\left(n-1\right)},$$

and (4.1) is proven. This proves Theorem 4.1.

What I find interesting is that Theorem 4.1 can be made stronger - the condition that $x_1, x_2, ..., x_n$ are nonnegative can be replaced by the weaker condition that $x_i < 1$ for every $i \in \{1, 2, ..., n\}$. The resulting fact is, however, more difficult to prove - see [4].

Now we come to the main result:

Theorem 4.2. Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ be $2n$ nonnegative reals. Then,

$$\sum_{i=1}^{n} \frac{a_i}{\sum\limits_{1\leq j\leq n,\ j\neq i} a_j} \sum_{1\leq j\leq n,\ j\neq i} b_j \geq \sqrt{\frac{2n}{n-1}\cdot \sum_{1\leq i<j\leq n} b_i b_j} \geq \frac{2n}{n-1}\cdot \frac{\sum\limits_{1\leq i<j\leq n} b_i b_j}{\sum\limits_{i=1}^{n} b_i}.$$

As a particular case of Theorem 4.2 (for $n = 3$, $a_1 = a$, $a_2 = b$, $a_3 = c$, $b_1 = u$, $b_2 = v$, $b_3 = w$), we obtain:

Theorem 4.3. If a, b, c, u, v, w are six nonnegative reals, then

$$\frac{a}{b+c}\left(v+w\right)+\frac{b}{c+a}\left(w+u\right)+\frac{c}{a+b}\left(u+v\right) \geq \sqrt{3\left(vw+wu+uv\right)} \geq \frac{3\left(vw+wu+uv\right)}{u+v+w}.$$

This inequality is a strengthening of the celebrated inequality

$$\frac{a}{b+c}\left(v+w\right)+\frac{b}{c+a}\left(w+u\right)+\frac{c}{a+b}\left(u+v\right) \geq \frac{3\left(vw+wu+uv\right)}{u+v+w},$$

which was proposed by Walther Janous as Crux Mathematicorum problem #1672, and discussed in [5] (among other places).

Proof of Theorem 4.2. WLOG assume that $a_1 + a_2 + ... + a_n = 1$. For every $i \in \{1, 2, ..., n\}$, denote

$$c_i = \frac{a_i}{\sum\limits_{1\leq j\leq n,\ j\neq i} a_j} = \frac{a_i}{\left(a_1+a_2+...+a_n\right)-a_i} = \frac{a_i}{1-a_i}.$$

Then,

$$\sum_{i=1}^{n} \frac{a_i}{\sum\limits_{1\leq j\leq n,\ j\neq i} a_j} \sum_{1\leq j\leq n,\ j\neq i} b_j = \sum_{i=1}^{n} c_i \sum_{1\leq j\leq n,\ j\neq i} b_j = \sum_{1\leq i\neq j\leq n} c_i b_j. \qquad (4.4)$$

But according to Theorem 1.1, we have

$$\left(\sum_{1\leq i\neq j\leq n} c_i b_j\right)^2 \geq 4 \sum_{1\leq i<j\leq n} c_i c_j \sum_{1\leq i<j\leq n} b_i b_j,$$

so that, after taking the square root,

$$\sum_{1\leq i\neq j\leq n} c_i b_j \geq 2\sqrt{\sum_{1\leq i<j\leq n} c_i c_j \sum_{1\leq i<j\leq n} b_i b_j}. \qquad (4.5)$$

But

$$\sum_{1\leq i<j\leq n} c_i c_j = \sum_{1\leq i<j\leq n} \frac{a_i}{1-a_i}\cdot\frac{a_j}{1-a_j} = \sum_{1\leq i<j\leq n} \frac{a_i a_j}{(1-a_i)(1-a_j)},$$

and Theorem 4.1 yields

$$\sum_{1\leq i<j\leq n} \frac{a_i a_j}{(1-a_i)(1-a_j)} \geq \frac{n}{2(n-1)}.$$

Hence,

$$\sum_{1\leq i<j\leq n} c_i c_j \geq \frac{n}{2(n-1)}.$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{n} \frac{a_i}{\sum\limits_{1\leq j\leq n,\ j\neq i} a_j} \sum_{1\leq j\leq n,\ j\neq i} b_j &= \sum_{1\leq i\neq j\leq n} c_i b_j \qquad \text{(by (4.4))}\\
&\geq 2\sqrt{\sum_{1\leq i<j\leq n} c_i c_j \sum_{1\leq i<j\leq n} b_i b_j} \qquad \text{(by (4.5))}\\
&\geq 2\sqrt{\frac{n}{2(n-1)}\cdot\sum_{1\leq i<j\leq n} b_i b_j} = \sqrt{\frac{2n}{n-1}\cdot\sum_{1\leq i<j\leq n} b_i b_j}.
\end{aligned}$$

Hence, it remains only to prove that

$$\sqrt{\frac{2n}{n-1}\cdot\sum_{1\leq i<j\leq n} b_i b_j} \geq \frac{2n}{n-1}\cdot\frac{\sum\limits_{1\leq i<j\leq n} b_i b_j}{\sum\limits_{i=1}^{n} b_i}.$$

Upon squaring, this becomes

$$\frac{2n}{n-1}\cdot\sum_{1\leq i<j\leq n} b_i b_j \geq \left(\frac{2n}{n-1}\cdot\frac{\sum\limits_{1\leq i<j\leq n} b_i b_j}{\sum\limits_{i=1}^{n} b_i}\right)^2,$$

and simplifies to

$$\left(\sum_{i=1}^{n} b_i\right)^2 \geq \frac{2n}{n-1}\cdot\sum_{1\leq i<j\leq n} b_i b_j.$$

But this is the inequality (4.3), applied to $u_1 = b_1$, $u_2 = b_2$, ..., $u_n = b_n$.

This completes the proof of Theorem 4.2.

5. Another application

As another consequence of Theorem 1.1, we can show:

Theorem 5.1. For any three reals a, b, c, we have

$$((b+c)\,bc+(c+a)\,ca+(a+b)\,ab)^2 \geq 4\,(bc+ca+ab)\left(b^2c^2+c^2a^2+a^2b^2\right).$$

Proof of Theorem 5.1. Applying Theorem 1.2 for $x=a^2$, $y=b^2$, $z=c^2$ (obviously, $yz+zx+xy\geq 0$ is satisfied since $yz+zx+xy=b^2c^2+c^2a^2+a^2b^2$), we get

$$\left(ab^2+ac^2+bc^2+ba^2+ca^2+cb^2\right) \geq 4\,(bc+ca+ab)\left(b^2c^2+c^2a^2+a^2b^2\right),$$

what rewrites as

$$((b+c)\,bc+(c+a)\,ca+(a+b)\,ab)^2 \geq 4\,(bc+ca+ab)\left(b^2c^2+c^2a^2+a^2b^2\right),$$

and Theorem 5.1 is proven.

Note that the particular case of Theorem 5.1 when the reals a, b, c are nonnegative was used as Lemma 3 in [6], post #2.

With the help of Theorem 5.1, the following result can be shown:

Theorem 5.2. Let a, b, c be three reals, no two of which are zero. Then,

$$\frac{a^2(b+c)^2}{b^2+c^2}+\frac{b^2(c+a)^2}{c^2+a^2}+\frac{c^2(a+b)^2}{a^2+b^2}\geq 2\,(bc+ca+ab).$$

Proof of Theorem 5.2. We have

$$\begin{aligned}\frac{a^2(b+c)^2}{b^2+c^2}+\frac{b^2(c+a)^2}{c^2+a^2}+\frac{c^2(a+b)^2}{a^2+b^2} &= \frac{\left(a^2(b+c)\right)^2}{a^2b^2+c^2a^2}+\frac{\left(b^2(c+a)\right)^2}{b^2c^2+a^2b^2}+\frac{\left(c^2(a+b)\right)^2}{c^2a^2+b^2c^2}\\ &\geq \frac{\left(a^2(b+c)+b^2(c+a)+c^2(a+b)\right)^2}{(a^2b^2+c^2a^2)+(b^2c^2+a^2b^2)+(c^2a^2+b^2c^2)}\end{aligned}$$

by the Cauchy-Schwarz inequality in the Engel form. Thus, it remains to prove that

$$\frac{\left(a^2(b+c)+b^2(c+a)+c^2(a+b)\right)^2}{(a^2b^2+c^2a^2)+(b^2c^2+a^2b^2)+(c^2a^2+b^2c^2)}\geq 2\,(bc+ca+ab).$$

This rewrites as

$$\frac{((b+c)\,bc+(c+a)\,ca+(a+b)\,ab)^2}{2\left(b^2c^2+c^2a^2+a^2b^2\right)}\geq 2\,(bc+ca+ab),$$

what simplifies to

$$((b+c)\,bc+(c+a)\,ca+(a+b)\,ab)^2 \geq 4\,(bc+ca+ab)\left(b^2c^2+c^2a^2+a^2b^2\right).$$

But this follows from Theorem 5.1. Thus, Theorem 5.2 is proved.

The particular case of Theorem 5.2 when the reals a, b, c are nonnegative is problem 7.8.1 in [3], where it is proven using the Sum of Squares (SOS) method.

6. An USA TST problem

Our final application of Theorem 1.1 will be problem 6 from the USA TST 2001, which has received some different solutions in [2]:

Theorem 6.1. Let a, b, c be three positive reals such that $a+b+c \geq abc$. Then, at least two of the three inequalities $\frac{2}{a}+\frac{3}{b}+\frac{6}{c} \geq 6$, $\frac{2}{b}+\frac{3}{c}+\frac{6}{a} \geq 6$ and $\frac{2}{c}+\frac{3}{a}+\frac{6}{b} \geq 6$ are true.

Proof of Theorem 6.1. Assume the contrary, i. e. assume that at most one of the three inequalities $\frac{2}{a}+\frac{3}{b}+\frac{6}{c} \geq 6$, $\frac{2}{b}+\frac{3}{c}+\frac{6}{a} \geq 6$ and $\frac{2}{c}+\frac{3}{a}+\frac{6}{b} \geq 6$ is true. Then, we can WLOG say that $\frac{2}{b}+\frac{3}{c}+\frac{6}{a} < 6$ and $\frac{2}{c}+\frac{3}{a}+\frac{6}{b} < 6$. But, applying Theorem 1.2 to the six reals 2, 3, 6, $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ (which surely satisfy $3\cdot 6+6\cdot 2+2\cdot 3 \geq 0$), we obtain

$$\left(2\cdot\frac{1}{b}+2\cdot\frac{1}{c}+3\cdot\frac{1}{c}+3\cdot\frac{1}{a}+6\cdot\frac{1}{a}+6\cdot\frac{1}{b}\right)^2 \geq 4\left(3\cdot 6+6\cdot 2+2\cdot 3\right)\left(\frac{1}{b}\cdot\frac{1}{c}+\frac{1}{c}\cdot\frac{1}{a}+\frac{1}{a}\cdot\frac{1}{b}\right).$$

In other words,

$$\left(\left(\frac{2}{b}+\frac{3}{c}+\frac{6}{a}\right)+\left(\frac{2}{c}+\frac{3}{a}+\frac{6}{b}\right)\right)^2 \geq 4\cdot 36\cdot\frac{a+b+c}{abc}.$$

Since $a+b+c \geq abc$, we have $\frac{a+b+c}{abc} \geq 1$, and thus this entails

$$\left(\left(\frac{2}{b}+\frac{3}{c}+\frac{6}{a}\right)+\left(\frac{2}{c}+\frac{3}{a}+\frac{6}{b}\right)\right)^2 \geq 4\cdot 36.$$

On the other hand, $\frac{2}{b}+\frac{3}{c}+\frac{6}{a} < 6$ and $\frac{2}{c}+\frac{3}{a}+\frac{6}{b} < 6$ imply

$$\left(\left(\frac{2}{b}+\frac{3}{c}+\frac{6}{a}\right)+\left(\frac{2}{c}+\frac{3}{a}+\frac{6}{b}\right)\right)^2 < (6+6)^2 = 4\cdot 36.$$

This is a contradiction. Hence, our assumption was wrong, and Theorem 6.1 is proved.

As a sidenote, a different proof of Theorem 6.1 can be obtained by showing that

$$\left(\frac{2}{b}+\frac{3}{c}+\frac{6}{a}\right)\left(\frac{2}{c}+\frac{3}{a}+\frac{6}{b}\right) \geq \left(3\cdot 6+6\cdot 2+2\cdot 3\right)\left(\frac{1}{b}\cdot\frac{1}{c}+\frac{1}{c}\cdot\frac{1}{a}+\frac{1}{a}\cdot\frac{1}{b}\right).$$

This follows from Theorem 10 **j)** in my note [7], applied to the six nonnegative reals 2, 3, 6, $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ (noting that 2, 3, 6 are the squares of the sidelengths of a triangle).

References

[1] Sunchips et al., *MathLinks topic #105871 ("6 variables").*
`http://www.mathlinks.ro/Forum/viewtopic.php?t=105871`

[2] Hxtung et al., *MathLinks topic #139 ("Inequaly: USA selection team").*
`http://www.mathlinks.ro/Forum/viewtopic.php?t=139`

[3] Vasile Cîrtoaje, *Algebraic Inequalities - Old and New Methods*, Gil 2006.

[4] Darij Grinberg, *Math Time problem proposal #1.*

[5] Harazi et al., *MathLinks topic #1688 ("Nice inequality comes back").*
`http://www.mathlinks.ro/Forum/viewtopic.php?t=1688`

[6] Pvthuan et al., *MathLinks topic #21679 ("easy or difficult").*
`http://www.mathlinks.ro/Forum/viewtopic.php?t=21679`

[7] Darij Grinberg, *The Vornicu-Schur inequality and its variations (MathLinks article).*
`http://www.mathlinks.ro/Forum/portal.php?t=162684`