The Vornicu-Schur inequality and its variations

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(version 13 August 2007)

The following is a note I have contributed to the Vietnamese inequality book

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A rather small part of this note has appeared in section 4.8 of this book, supplemented with applications. What follows is the original note (slightly edited), focusing rather on variations and possible extensions of the Vornicu-Schur inequality than on applications. There are more such variations than you might think, and most of these are rarely applied because of their low renownedness. I hope that it will be a source of inspiration to find new results and new approaches to older inequalities.

The Schur inequality, as well as some of its generalizations subsumed under the name "Vornicu-Schur inequality", are known to have some efficient and unexpected applications to proving inequalities. Here we are going to discuss some known ideas in this field and some possible ways to proceed further.

First a convention: In the following, we will use the sign $\sum$ for cyclic sums. This means, in particular, that if a, b, c are three variables, and x, y, z are three variables "corresponding" to a, b, c, and if f is a function of six variables, then

$$\sum f\left(a,b,c,x,y,z\right) = f\left(a,b,c,x,y,z\right) + f\left(b,c,a,y,z,x\right) + f\left(c,a,b,z,x,y\right).$$

For instance,

$$\sum \frac{a}{b}x^2 z = \frac{a}{b}x^2 z + \frac{b}{c}y^2 x + \frac{c}{a}z^2 y.$$

1. The Vornicu-Schur inequality

There are different things referred to as Vornicu-Schur inequality; here is a possible collection of results:

Theorem 1 (Vornicu-Schur inequality, more properly called Vornicu-Schur-Mildorf inequality, or just generalized Schur inequality). Let a, b, c be three reals, and let x, y, z be three nonnegative reals. Then, the inequality

$$x\left(a-b\right)\left(a-c\right) + y\left(b-c\right)\left(b-a\right) + z\left(c-a\right)\left(c-b\right) \geq 0$$

holds if one of the following (sufficient) conditions is fulfilled:

a) We have $a \geq b \geq c$ and $x \geq y$.

b) We have $a \geq b \geq c$ and $z \geq y$.

c) We have $a \geq b \geq c$ and $x + z \geq y$.

d) The reals a, b, c are nonnegative and satisfy $a \geq b \geq c$ and $ax \geq by$.

e) The reals a, b, c are nonnegative and satisfy $a \geq b \geq c$ and $cz \geq by$.

f) The reals a, b, c are nonnegative and satisfy $a \geq b \geq c$ and $ax + cz \geq by$.

g) The reals x, y, z are sidelengths of a triangle.

h) The reals x, y, z are the squares of the sidelengths of a triangle.

i) The reals ax, by, cz are sidelengths of a triangle.

j) The reals ax, by, cz are the squares of the sidelengths of a triangle.

k) There exists a convex function $t : I \to \mathbb{R}^+$, where I is an interval containing the reals a, b, c, such that $x = t(a)$, $y = t(b)$, $z = t(c)$.

Hereby, we denote by $\mathbb{R}^+$ the set of all *nonnegative* reals.

Proof of Theorem 1. It is clear that Theorem 1 **a)** and **b)** follow from Theorem 1 **c)** (since each of the inequalities $x \geq y$ and $z \geq y$ yields $x + z \geq y$), and that Theorem 1 **d)** and **e)** follow from Theorem 1 **f)** (since each of the inequalities $ax \geq by$ and $cz \geq by$ implies $ax + cz \geq by$, as the reals a, b, c are nonnegative). Hence, we won't have to give separate proofs for Theorem 1 **a)**, **b)**, **d)**, **e)**.

c) Since $a - b \geq 0$ and $a - c \geq b - c$ (both because of $a \geq b \geq c$), we have

$$x(a-b)(a-c) \geq x(a-b)(b-c) = -x(b-c)(b-a).$$

Since $c - a \leq b - a$ and $c - b \leq 0$ (again since $a \geq b \geq c$), we have

$$z(c-a)(c-b) \geq z(b-a)(c-b) = -z(b-c)(b-a).$$

Hence,

$$\begin{aligned}
&x(a-b)(a-c) + y(b-c)(b-a) + z(c-a)(c-b) \\
&\geq -x(b-c)(b-a) + y(b-c)(b-a) - z(b-c)(b-a) \\
&= (y - (x+z))(b-c)(b-a) = ((x+z) - y)(b-c)(a-b).
\end{aligned}$$

Since $(x+z) - y \geq 0$ (since $x + z \geq y$) and $b - c \geq 0$ and $a - b \geq 0$ (since $a \geq b \geq c$), it follows that

$$x(a-b)(a-c) + y(b-c)(b-a) + z(c-a)(c-b) \geq 0.$$

This proves Theorem 1 **c)**.

f) Since the reals cz, by, ax are nonnegative and we have $ab \geq ca \geq bc$ (since $a \geq b \geq c$) and $cz + ax \geq by$, we can apply Theorem 1 **c)** to the reals ab, ca, bc, cz, by, ax instead of a, b, c, x, y, z, respectively. We obtain

$$\begin{aligned}
cz(ab-ca)(ab-bc) + by(ca-bc)(ca-ab) + ax(bc-ab)(bc-ca) &\geq 0, && \text{or, equivalently,} \\
abcz(c-a)(c-b) + abcy(b-c)(b-a) + abcx(a-b)(a-c) &\geq 0, && \text{or, equivalently,} \\
z(c-a)(c-b) + y(b-c)(b-a) + x(a-b)(a-c) &\geq 0,
\end{aligned}$$

and Theorem 1 **f)** is proven.

g) This follows from Theorem 1 **c)**, because we can WLOG assume that $a \geq b \geq c$ and then have $x + z \geq y$ by the triangle inequality, because x, y, z are the sidelengths of a triangle.

Theorem 1 **g)** also follows from **h)** with a little work, but let us also give an independent proof for Theorem 1 **g)**: Since x, y, z are the sidelengths of a triangle, by the triangle inequalities we have $y + z > x$, $z + x > y$, $x + y > z$, so that $y + z - x > 0$,

$z+x-y>0$, $x+y-z>0$. Thus,

$$\begin{aligned}
&x(a-b)(a-c)+y(b-c)(b-a)+z(c-a)(c-b)=\sum x(a-b)(a-c)\\
&=\sum \frac{1}{2}((z+x-y)+(x+y-z))(a-b)(a-c)\\
&=\frac{1}{2}\left(\sum (z+x-y)(a-b)(a-c)+\sum (x+y-z)(a-b)(a-c)\right)\\
&=\frac{1}{2}\left(\sum (y+z-x)(c-a)(c-b)+\sum (y+z-x)(b-c)(b-a)\right)\\
&=\frac{1}{2}\sum (y+z-x)((c-a)(c-b)+(b-c)(b-a))=\frac{1}{2}\sum (y+z-x)(b-c)^2\geq 0,
\end{aligned}$$

and Theorem 1 **g)** is proven.

h) We have to prove that

$$u^2(a-b)(a-c)+v^2(b-c)(b-a)+w^2(c-a)(c-b)\geq 0,$$

where u, v, w are the sidelengths of a triangle.

Since u, v, w are the sidelengths of a triangle, the triangle inequality yields $v<u+w$, so that $v^2<(u+w)^2$. We can WLOG assume that $a\geq b\geq c$; then, $b-c\geq 0$ and $b-a\leq 0$, so that $(b-c)(b-a)\leq 0$. Thus, $v^2<(u+w)^2$ becomes $v^2(b-c)(b-a)\geq (u+w)^2(b-c)(b-a)$. Hence,

$$\begin{aligned}
&u^2(a-b)(a-c)+v^2(b-c)(b-a)+w^2(c-a)(c-b)\\
&\geq u^2(a-b)(a-c)+(u+w)^2(b-c)(b-a)+w^2(c-a)(c-b)\\
&=u^2(a-b)(a-c)+\left(u^2(b-c)(b-a)+2uw(b-c)(b-a)+w^2(b-c)(b-a)\right)\\
&+w^2(c-a)(c-b)\\
&=u^2((a-b)(a-c)+(b-c)(b-a))+2uw(b-c)(b-a)\\
&+w^2((b-c)(b-a)+(c-a)(c-b))\\
&=u^2(b-a)^2+2\cdot u(b-a)\cdot w(b-c)+w^2(b-c)^2=(u(b-a)+w(b-c))^2\geq 0,
\end{aligned}$$

and Theorem 1 **h)** is proven.

i) Since the reals ax, by, cz are sidelengths of a triangle and therefore positive, and x, y, z are nonnegative, it follows that a, b, c are positive. Now, we can obtain Theorem 1 **i)** by applying Theorem 1 **g)** to the reals ab, ca, bc, cz, by, ax instead of a, b, c, x, y, z, respectively (the exact proof is similar to the proof of Theorem 1 **f)**).

j) Since the reals ax, by, cz are squares of the sidelengths of a triangle and therefore positive, and x, y, z are nonnegative, it follows that a, b, c are positive. Now, we can obtain Theorem 1 **j)** by applying Theorem 1 **h)** to the reals ab, ca, bc, cz, by, ax instead of a, b, c, x, y, z, respectively (the exact proof is similar to the proof of Theorem 1 **f)**).

k) If two of the reals a, b, c are equal, then everything is trivial, so we can assume that the reals a, b, c are pairwisely distinct. WLOG assume that $a>b>c$. Then, $b-c>0$ and $a-b>0$, and

$$b=\frac{ba-bc}{a-c}=\frac{(b-c)a+(a-b)c}{a-c}=\frac{(b-c)a+(a-b)c}{(b-c)+(a-b)}.$$

Since $b-c>0$ and $a-b>0$, and since the function t is convex on the interval I containing the reals a, b, c, we have

$$t(b)=t\left(\frac{(b-c)a+(a-b)c}{(b-c)+(a-b)}\right)\leq\frac{(b-c)t(a)+(a-b)t(c)}{(b-c)+(a-b)}.$$

But if we had $t(b)>t(a)$ and $t(b)>t(c)$, then we would have

$$t(b)=\frac{(b-c)t(b)+(a-b)t(b)}{(b-c)+(a-b)}>\frac{(b-c)t(a)+(a-b)t(c)}{(b-c)+(a-b)}.$$

Hence, the inequalities $t(b)>t(a)$ and $t(b)>t(c)$ cannot both hold; hence, either $t(a)\geq t(b)$ or $t(c)\geq t(b)$. Since $x=t(a)$, $y=t(b)$, $z=t(c)$, this becomes: Either $x\geq y$ or $z\geq y$. If $x\geq y$, we can apply Theorem 1 **a)**; if $z\geq y$, we can apply Theorem 1 **b)**. Thus, Theorem 1 **k)** is proven.

The Vornicu-Schur inequality is also called the generalized Schur inequality, since the actual Schur inequality easily follows from it:

Theorem 2 (Schur inequality). a) Let a, b, c be three positive reals, and r a real. Then, the inequality

$$a^r(a-b)(a-c)+b^r(b-c)(b-a)+c^r(c-a)(c-b)\geq 0$$

holds.

b) Let a, b, c be three reals, and r an even nonnegative integer. Then, the inequality

$$a^r(a-b)(a-c)+b^r(b-c)(b-a)+c^r(c-a)(c-b)\geq 0$$

holds.

Proof of Theorem 2. **a)** We can WLOG assume that $a\geq b\geq c$. If $r\geq 0$, then $a\geq b$ yields $a^r\geq b^r$, so the inequality in question follows from Theorem 1 **a)**; if $r\leq 0$, then $b\geq c$ yields $c^r\geq b^r$, so the inequality in question follows from Theorem 1 **b)**.

b) This follows from Theorem 1 **k)**, applied to the convex function $t:\mathbb{R}\to\mathbb{R}^+$ defined by $t(u)=u^r$.

The main strength of the Vornicu-Schur inequality is that the various criteria cover a lot of different cases. When you face a (true) inequality of the type $\sum x(a-b)(a-c)\geq 0$, chances are high that the terms x, y, z satisfy one of the criteria of Theorem 1. Even apparent curiousities like Theorem 1 **h)** can be of use:

Exercise. Prove that for any three nonnegative reals a, b, c, we have

$$\left(b^2-c^2\right)^2+\left(c^2-a^2\right)^2+\left(a^2-b^2\right)^2\geq 4(b-c)(c-a)(a-b)(a+b+c).$$

Hint. Rewrite this inequality in the form $\sum(a+b)^2(a-b)(a-c)\geq 0$.

Exercise. Prove that if a, b, c are sidelengths of a triangle, then $a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0$. [This is IMO 1983 problem 6.]

Hint. Rewrite this as $\sum c(a+b-c)(a-b)(a-c)\geq 0$. Now, this follows from Theorem 1 **g)** once it is shown that $c(a+b-c)$, $a(b+c-a)$, $b(c+a-b)$ are the sidelengths of a triangle.

Exercise. Prove that if a, b, c are sidelengths of a triangle, and $t\geq 1$ is a real, then

$$(t+1)\sum a^3b\geq\sum ab^3+tabc(a+b+c).$$

Hint. This inequality was posted on MathLinks and received pretty ugly proofs. But we can do better: First, show that it is enough to prove it for $t = 1$; now, for $t = 1$, it becomes $\sum b(b+a)(a-b)(a-c) \geq 0$, what follows from $\sum b^2(a-b)(a-c) \geq 0$ and $\sum ba(a-b)(a-c) \geq 0$. The first of these inequalities follows from Theorem 1 **h)**; the second one is equivalent to $\sum a^2 b(a-b) \geq 0$, what is the previous exercise.

2. A generalization with odd functions

There is a simple generalization of parts **a)**, **b)**, **c)**, **g)**, **k)** of Theorem 1:

Theorem 3. Let $J \subseteq \mathbb{R}$ be an interval, and let $p : J \to \mathbb{R}$ be an odd, monotonically increasing function such that $p(t) \geq 0$ for all nonnegative $t \in J$ and $p(t) \leq 0$ for all nonpositive $t \in J$. Let a, b, c be three reals, and let x, y, z be three nonnegative reals. Then, the inequality

$$x \cdot p(a-b) \cdot p(a-c) + y \cdot p(b-c) \cdot p(b-a) + z \cdot p(c-a) \cdot p(c-b) \geq 0$$

holds if the numbers $a-b$, $a-c$, $b-c$, $b-a$, $c-a$, $c-b$ lie in the interval J and one of the following (sufficient) conditions is fulfilled:

a) We have $a \geq b \geq c$ and $x \geq y$.

b) We have $a \geq b \geq c$ and $z \geq y$.

c) We have $a \geq b \geq c$ and $x + z \geq y$.

d) The reals x, y, z are sidelengths of a triangle.

e) There exists a convex function $t : I \to \mathbb{R}^+$, where I is an interval containing the reals a, b, c, such that $x = t(a)$, $y = t(b)$, $z = t(c)$.

The *proof* of this theorem is analogous to the proofs of the respective parts of Theorem 1 (for the proof of Theorem 3 **d)**, we have to apply Theorem 3 **c)**).

As a particular case of Theorem 3, the following result (due to Mildorf?) can be obtained:

Theorem 4. Let k be a nonnegative integer. Let a, b, c be three reals, and let x, y, z be three nonnegative reals. Then, the inequality

$$x(a-b)^k (a-c)^k + y(b-c)^k (b-a)^k + z(c-a)^k (c-b)^k \geq 0$$

holds if one of the following (sufficient) conditions is fulfilled:

a) We have $a \geq b \geq c$ and $x \geq y$.

b) We have $a \geq b \geq c$ and $z \geq y$.

c) We have $a \geq b \geq c$ and $x + z \geq y$.

d) The reals x, y, z are sidelengths of a triangle.

e) There exists a convex function $t : I \to \mathbb{R}^+$, where I is an interval containing the reals a, b, c, such that $x = t(a)$, $y = t(b)$, $z = t(c)$.

Proof of Theorem 4. If k is even, then k-th powers are always nonnegative and thus the inequality in question is obviously true. If k is odd, then the function $p : \mathbb{R} \to \mathbb{R}$ defined by $p(t) = t^k$ is an odd, monotonically increasing function such that $p(t) \geq 0$ for all nonnegative $t \in \mathbb{R}$ and $p(t) \leq 0$ for all nonpositive $t \in \mathbb{R}$. And thus, the assertion of Theorem 4 follows from Theorem 3.

Exercise. a) For any triangle ABC, prove the inequality

$$x \sin\frac{A-B}{2} \sin\frac{A-C}{2} + y \sin\frac{B-C}{2} \sin\frac{B-A}{2} + z \sin\frac{C-A}{2} \sin\frac{C-B}{2} \geq 0$$

aa) for $x = y = z = 1$.

ab) for $x = \sin A'$, $y = \sin B'$, $z = \sin C'$, where A', B', C' are the angles of another triangle $A'B'C'$.

ac) for $x = \sin \frac{A}{2}$, $y = \sin \frac{B}{2}$, $z = \sin \frac{C}{2}$.

b) For an acute-angled triangle ABC, prove the inequality

$$x \sin (A - B) \sin (A - C) + y \sin (B - C) \sin (B - A) + z \sin (C - A) \sin (C - B) \geq 0$$

ba), bb), bc), for the same values of x, y, z as in **a)**.

bd) for $x = \cos A$, $y = \cos B$, $z = \cos C$.

c) Prove **ba)**, **bb)**, **bc)** not only for an acute-angled triangle ABC, but also for a triangle ABC all of whose angles are $\leq 120^\circ$.

d) Create some more inequalities of this kind.

3. The convexity approach

All the conditions given in Theorem 1 for the inequality $\sum x (a - b) (a - c) \geq 0$ to hold are just sufficient, none of them is necessary. We could wonder how a necessary and sufficient condition looks like. The answer is given by the following result:

Theorem 5. Let a, b, c, x, y, z be six reals.

a) The inequality

$$x (a - b) (a - c) + y (b - c) (b - a) + z (c - a) (c - b) \geq 0$$

holds if we can find a convex function $u : I \to \mathbb{R}$, where I is an interval containing the reals a, b, c, which satisfies $x = (b - c)^2 u (a)$, $y = (c - a)^2 u (b)$, $z = (a - b)^2 u (c)$.

b) The inequality

$$x (a - b) (a - c) + y (b - c) (b - a) + z (c - a) (c - b) \leq 0$$

holds if we can find a concave function $u : I \to \mathbb{R}$, where I is an interval containing the reals a, b, c, which satisfies $x = (b - c)^2 u (a)$, $y = (c - a)^2 u (b)$, $z = (a - b)^2 u (c)$.

c) If the reals a, b, c are pairwisely distinct, then the "if" in parts **a)** and **b)** can be replaced by "if and only if".

Proof of Theorem 5. **a)** If two of the reals a, b, c are equal, then everything is trivial, so we can assume that the reals a, b, c are pairwisely distinct. We can WLOG assume that $a > b > c$.

The inequality in question, $\sum x (a - b) (a - c) \geq 0$, rewrites as

$$\sum (b - c)^2 u (a) \cdot (a - b) (a - c) \geq 0, \qquad \text{what is equivalent to}$$

$$\sum (b - c)^2 u (a) \cdot (- (a - b) (c - a)) \geq 0, \qquad \text{what is equivalent to}$$

$$- (b - c) (c - a) (a - b) \cdot \sum (b - c) u (a) \geq 0.$$

Since $a > b > c$, we have $b - c > 0$, $c - a < 0$, $a - b > 0$, so that $- (b - c) (c - a) (a - b) > 0$, so this inequality is equivalent to $\sum (b - c) u (a) \geq 0$.

Now,

$$b = \frac{ba - bc}{a - c} = \frac{(b - c) a + (a - b) c}{a - c} = \frac{(b - c) a + (a - b) c}{(b - c) + (a - b)}.$$

Since $b-c>0$ and $a-b>0$, and since the function u is convex on the interval I containing the reals a, b, c, we have

$$u(b)=u\left(\frac{(b-c)\,a+(a-b)\,c}{(b-c)+(a-b)}\right)\leq\frac{(b-c)\,u(a)+(a-b)\,u(c)}{(b-c)+(a-b)},$$

so that

$$(b-c)\,u(a)+(a-b)\,u(c)\geq((b-c)+(a-b))\,u(b)=-(c-a)\,u(b)\,,$$

so that $(b-c)\,u(a)+(c-a)\,u(b)+(a-b)\,u(c)\geq 0$, or, equivalently, $\sum(b-c)\,u(a)\geq 0$. Theorem 5 **a)** is proven.

The proof of Theorem 5 **b)** is analogous to the proof of Theorem 5 **a)**, and the proof of Theorem 5 **c)** is left to the reader.

One of the consequences of Theorem 5 is:

Theorem 6. Let a, b, c be three positive reals, and r a real. Then, for $r\geq 1$ and for $r\leq 0$ we have

$$a^r\,(b-c)^2\,(a-b)\,(a-c)+b^r\,(c-a)^2\,(b-c)\,(b-a)+c^r\,(a-b)^2\,(c-a)\,(c-b)\geq 0,$$

while for $0\leq r\leq 1$ we have

$$a^r\,(b-c)^2\,(a-b)\,(a-c)+b^r\,(c-a)^2\,(b-c)\,(b-a)+c^r\,(a-b)^2\,(c-a)\,(c-b)\leq 0.$$

Proof of Theorem 6. Apply Theorem 5 to $x=(b-c)^2\,u(a)\,,\ y=(c-a)^2\,u(b)\,,\ z=(a-b)^2\,u(c)\,,$ where $u:\mathbb{R}^+\to\mathbb{R}^+$ is the function defined by $u(t)=t^r$. For $r\geq 1$ and for $r\leq 0$, the function u is convex, so we can apply Theorem 5 **a)**; for $0\leq r\leq 1$, the function u is concave and we must apply Theorem 5 **b)**.

Note that Theorem 6 for $r\leq 0$ has been stated by Spanferkel on MathLinks.

4. Vornicu-Schur and SOS

In the proof of Theorem 1 **g)**, we noticed the identity

$$\sum x\,(a-b)\,(a-c)=\frac{1}{2}\sum(y+z-x)\,(b-c)^2\,.$$

This identity relates the Vornicu-Schur inequality to the SOS principle. Indeed, Theorems 1 **a)**, **b)**, **c)**, **g)** can be easily proven using SOS with this identity. Therefore, a number of proofs done using Vornicu-Schur can be rewritten using SOS, and vice versa.

Exercise. Prove using Theorem 1 **c)** (though this is overkill) the following inequality: If a, b, c, x, y, z are six nonnegative reals satisfying $a\geq b\geq c$ and $x\leq y\leq z$, then

$$x\,(b-c)^2\,(b+c-a)+y\,(c-a)^2\,(c+a-b)+z\,(a-b)^2\,(a+b-c)\geq 0.$$

5. One more degree of freedom

Here is an example of how certain criteria for $\sum x\,(a-b)\,(a-c) \geq 0$ can be extended to more free parameters:

Theorem 7. Let a, b, c, x, y, z be six nonnegative reals, and u and v two reals.

a) If $u \leq 1$ and $v \leq 1$, and the number arrays $(a,\ b,\ c)$ and $(x,\ y,\ z)$ are equally sorted, then

$$x\,(a-ub)\,(a-vc) + y\,(b-uc)\,(b-va) + z\,(c-ua)\,(c-vb) \geq 0.$$

b) If $u \geq 1$ and $v \geq 1$, and the number arrays $(a,\ b,\ c)$ and $(x,\ y,\ z)$ are oppositely sorted, then

$$x\,(a-ub)\,(a-vc) + y\,(b-uc)\,(b-va) + z\,(c-ua)\,(c-vb) \geq 0.$$

Proof of Theorem 7. **a)** We have to prove that $\sum x\,(a-ub)\,(a-vc) \geq 0$. Since $u \leq 1$, we have $u = 1-U$ for some real $U \geq 0$, and similarly $v = 1-V$ for some real $V \geq 0$. Thus,

$$\sum x\,(a-ub)\,(a-vc) = \sum x\,(a-(1-U)\,b)\,(a-(1-V)\,c) = \sum x\,((a-b)+Ub)\,((a-c)+Vc)$$
$$= \sum \left(x\,(a-b)\,(a-c) + x\,(a-b)\,Vc + xUb\,(a-c) + xUbVc\right)$$
$$= \sum \left(x\,(a-b)\,(a-c) + Vx\,(a-b)\,c + Ux\,(a-c)\,b + UVxbc\right)$$
$$= \sum x\,(a-b)\,(a-c) + V\sum x\,(a-b)\,c + U\sum x\,(a-c)\,b + UV\sum xbc.$$

Since the number arrays $(a,\ b,\ c)$ and $(x,\ y,\ z)$ are equally sorted, we can WLOG assume that $a \geq b \geq c$ and then have $x \geq y$, so that Theorem 1 **a)** yields $\sum x\,(a-b)\,(a-c) \geq 0$. Also, trivially, $\sum xbc \geq 0$. Hence, in order to prove that $\sum x\,(a-ub)\,(a-vc) \geq 0$, it will be enough to show that $\sum x\,(a-b)\,c \geq 0$ and $\sum x\,(a-c)\,b \geq 0$. But $\sum x\,(a-b)\,c = \sum x\,(ca-bc) = \sum x \cdot ca - \sum x \cdot bc \geq 0$, because $\sum x \cdot ca \geq \sum x \cdot bc$ by the rearrangement inequality (in fact, since the number arrays $(a,\ b,\ c)$ and $(x,\ y,\ z)$ are equally sorted, and the number arrays $(a,\ b,\ c)$ and $(bc,\ ca,\ ab)$ are oppositely sorted, it follows that the number arrays $(bc,\ ca,\ ab)$ and $(x,\ y,\ z)$ are oppositely sorted, so that $\sum x \cdot ca \geq \sum x \cdot bc$). Similarly, $\sum x\,(a-c)\,b \geq 0$. Hence, the proof of Theorem 7 **a)** is complete.

The proof of Theorem 7 **b)** is analogous.

Note that about certain particular cases, more can be said:

Exercise. If a, b, c are three nonnegative reals, and k a real, then prove that

$$\frac{(a-kb)\,(a-kc)}{a} + \frac{(b-kc)\,(b-ka)}{b} + \frac{(c-ka)\,(c-kb)}{c} \geq 0.$$

6. More variables?

A very natural question to ask is whether the Vornicu-Schur inequality (with any of its criteria) can be extended to more than three variables. Few is known about this yet. The first result of this kind is probably problem 1 of the IMO 1971:

Exercise. **a)** Let n be a positive integer. Show that the inequality

$$\sum_{i=1}^{n} \prod_{1 \leq j \leq n;\ j \neq i} (a_i - a_j) \geq 0$$

holds for every n reals a_1, a_2, ..., a_n if and only if $n = 2$, $n = 3$ or $n = 5$.

b) Why doesn't this change if we replace "reals" by "positive reals"?

Apart from this, I have not seen any notable success in extending Vornicu-Schur to several variables. Any new result would be a considerable progress here, since there are few methods known to prove inequalities for several variables and a generalized Schur inequality will be a new one.

7. Sum · Sum ≥ Sum · Sum inequalities

Here is a quite useful consequence of Theorem 1:

Theorem 8. Let a, b, c be three reals, and let x, y, z be three nonnegative reals. Then, the inequality

$$(azx + bxy + cyz)(axy + byz + czx) \geq (yz + zx + xy)(bcyz + cazx + abxy)$$

holds if one of the following (sufficient) conditions is fulfilled:

a) We have $a \geq b \geq c$ and $x \geq y$.

b) We have $a \geq b \geq c$ and $z \geq y$.

c) We have $a \geq b \geq c$ and $x + z \geq y$.

d) The reals a, b, c are nonnegative and satisfy $a \geq b \geq c$ and $ax \geq by$.

e) The reals a, b, c are nonnegative and satisfy $a \geq b \geq c$ and $cz \geq by$.

f) The reals a, b, c are nonnegative and satisfy $a \geq b \geq c$ and $ax + cz \geq by$.

g) The reals x, y, z are sidelengths of a triangle.

h) The reals x, y, z are the squares of the sidelengths of a triangle.

i) The reals ax, by, cz are sidelengths of a triangle.

j) The reals ax, by, cz are the squares of the sidelengths of a triangle.

k) There exists a convex function $t : I \to \mathbb{R}^+$, where I is an interval containing the reals a, b, c, such that $x = t(a)$, $y = t(b)$, $z = t(c)$.

Proof of Theorem 8. According to Theorem 1, each of the conditions **a)**, **b)**, **c)**, **d)**, **e)**, **f)**, **g)**, **h)**, **i)**, **j)**, **k)** yields

$$x(a-b)(a-c) + y(b-c)(b-a) + z(c-a)(c-b) \geq 0.$$

Now,

$$\begin{aligned}
&(azx + bxy + cyz)(axy + byz + czx) - (yz + zx + xy)(bcyz + cazx + abxy) \\
&= \sum azx \cdot axy + \sum azx \cdot byz + \sum azx \cdot czx - \sum yz \cdot bcyz - \sum yz \cdot cazx - \sum yz \cdot abxy \\
&= xyz \sum xa^2 + xyz \sum zab + \sum z^2x^2ca - \sum y^2z^2bc - xyz \sum zca - xyz \sum yab \\
&= xyz \sum xa^2 + xyz \sum xbc + \sum y^2z^2bc - \sum y^2z^2bc - xyz \sum xab - xyz \sum xca \\
&= xyz \sum \left(xa^2 + xbc - xab - xca\right) = xyz \sum x(a-b)(a-c) \\
&= \underbrace{xyz}_{\geq 0,\text{ since } x,\ y,\ z \text{ are nonnegative}} \underbrace{(x(a-b)(a-c) + y(b-c)(b-a) + z(c-a)(c-b))}_{\geq 0},
\end{aligned}$$

and thus

$$(azx + bxy + cyz)(axy + byz + czx) \geq (yz + zx + xy)(bcyz + cazx + abxy).$$

This proves Theorem 8.

Often, Theorem 8 is easier to apply in the following form:

Theorem 9. Let a, b, c be three reals, and let x, y, z be three nonnegative reals. Then, the inequality

$$(ay + bz + cx)(az + bx + cy) \geq (x + y + z)(xbc + yca + zab)$$

holds if one of the following (sufficient) conditions is fulfilled:

a) We have $a \geq b \geq c$ and $x \leq y$.

b) We have $a \geq b \geq c$ and $z \leq y$.

c) We have $a \geq b \geq c$ and $yz + xy \geq zx$.

d) The reals a, b, c are nonnegative and satisfy $a \geq b \geq c$ and $ay \geq bx$.

e) The reals a, b, c are nonnegative and satisfy $a \geq b \geq c$ and $cy \geq bz$.

f) The reals a, b, c are nonnegative and satisfy $a \geq b \geq c$ and $ayz + cxy \geq bzx$.

g) The reals yz, zx, xy are sidelengths of a triangle.

h) The reals yz, zx, xy are the squares of the sidelengths of a triangle.

i) The reals ayz, bzx, cxy are sidelengths of a triangle.

j) The reals ayz, bzx, cxy are the squares of the sidelengths of a triangle.

k) There exists a convex function $t : I \to \mathbb{R}^+$, where I is an interval containing the reals a, b, c, such that $yz = t(a)$, $zx = t(b)$, $xy = t(c)$.

Proof of Theorem 9. The conditions **a)**, **b)**, **c)**, **d)**, **e)**, **f)**, **g)**, **h)**, **i)**, **j)**, **k)** of Theorem 9 are equivalent to the conditions **a)**, **b)**, **c)**, **d)**, **e)**, **f)**, **g)**, **h)**, **i)**, **j)**, **k)** of Theorem 8 for the reals yz, zx, xy instead of x, y, z. Hence, if one of the conditions **a)**, **b)**, **c)**, **d)**, **e)**, **f)**, **g)**, **h)**, **i)**, **j)**, **k)** of Theorem 9 is fulfilled, then we can apply Theorem 8 to the reals yz, zx, xy instead of x, y, z, and we obtain

$$\begin{aligned}&(a \cdot xy \cdot yz + b \cdot yz \cdot zx + c \cdot zx \cdot xy)(a \cdot yz \cdot zx + b \cdot zx \cdot xy + c \cdot xy \cdot yz) \\ &\geq (zx \cdot xy + xy \cdot yz + yz \cdot zx)(bc \cdot zx \cdot xy + ca \cdot xy \cdot yz + ab \cdot yz \cdot zx).\end{aligned}$$

This simplifies to

$$x^2y^2z^2 (ay + bz + cx)(az + bx + cy) \geq x^2y^2z^2 (x + y + z)(xbc + yca + zab).$$

Hence, we obtain

$$(ay + bz + cx)(az + bx + cy) \geq (x + y + z)(xbc + yca + zab)$$

unless one of the reals x, y, z is 0. However, in the case when one of the reals x, y, z is 0, Theorem 9 is pretty easy to prove. Thus, Theorem 9 is completely proven.

Another useful corollary of Theorem 8 is:

Theorem 10. Let a, b, c, x, y, z be six nonnegative reals. Then, the inequality

$$(ay + bz + cx)(az + bx + cy) \geq (yz + zx + xy)(bc + ca + ab)$$

holds if one of the following (sufficient) conditions is fulfilled:

a) We have $\frac{x}{a} \geq \frac{y}{b} \geq \frac{z}{c}$ and $x \geq y$.

b) We have $\frac{x}{a} \geq \frac{y}{b} \geq \frac{z}{c}$ and $z \geq y$.

c) We have $\frac{x}{a} \geq \frac{y}{b} \geq \frac{z}{c}$ and $x + z \geq y$.

d) We have $\frac{x}{a} \geq \frac{y}{b} \geq \frac{z}{c}$ and $a \geq b$.

e) We have $\frac{x}{a} \geq \frac{y}{b} \geq \frac{z}{c}$ and $c \geq b$.

f) We have $\frac{x}{a} \geq \frac{y}{b} \geq \frac{z}{c}$ and $a + c \geq b$.

g) The reals x, y, z are sidelengths of a triangle.

h) The reals x, y, z are the squares of the sidelengths of a triangle.

i) The reals a, b, c are sidelengths of a triangle.

j) The reals a, b, c are the squares of the sidelengths of a triangle.

Proof of Theorem 10. Applying Theorem 8 to the nonnegative reals $\frac{x}{a}$, $\frac{y}{b}$, $\frac{z}{c}$, a, b, c instead of a, b, c, x, y, z, respectively, we see that, if these reals $\frac{x}{a}$, $\frac{y}{b}$, $\frac{z}{c}$, a, b, c satisfy one of the conditions **a)**, **b)**, **c)**, **d)**, **e)**, **f)**, **g)**, **h)**, **i)**, **j)**, **k)** of Theorem 8, the inequality

$$\begin{aligned}&\left(\frac{x}{a}ca + \frac{y}{b}ab + \frac{z}{c}bc\right)\left(\frac{x}{a}ab + \frac{y}{b}bc + \frac{z}{c}ca\right)\\ &\geq (bc + ca + ab)\left(\frac{y}{b}\cdot\frac{z}{c}\cdot bc + \frac{z}{c}\cdot\frac{x}{a}\cdot ca + \frac{x}{a}\cdot\frac{y}{b}\cdot ab\right)\end{aligned}$$

holds. This inequality simplifies to

$$(ay + bz + cx)(az + bx + cy) \geq (yz + zx + xy)(bc + ca + ab).$$

This is exactly the inequality claimed by Theorem 10. Now, the conditions **a)**, **b)**, **c)**, **g)**, **h)** of Theorem 8, applied to the nonnegative reals $\frac{x}{a}$, $\frac{y}{b}$, $\frac{z}{c}$, a, b, c instead of a, b, c, x, y, z, respectively, are equivalent to the conditions **d)**, **e)**, **f)**, **i)**, **j)** of Theorem 10 for the reals a, b, c, x, y, z. Hence, Theorem 10 **d)**, **e)**, **f)**, **i)**, **j)** is proven. In order to prove Theorem 10 **a)**, **b)**, **c)**, **g)**, **h)**, we apply Theorem 10 **e)**, **d)**, **f)**, **i)**, **j)** to the nonnegative reals z, y, x, c, b, a instead of the reals a, b, c, x, y, z, respectively. Thus, Theorem 10 is completely proven.

Exercise. Prove that, if we replace "six nonnegative reals" by "six reals" in Theorem 10, then Theorem 10 **g)**, **h)**, **i)**, **j)** will still remain true. [Note that Theorem 10 **j)** for arbitrary reals is problem 2.2.23 in Vasile Cîrtoaje, *Algebraic Inequalities - Old and New Methods*, Gil 2006.]