Mathematical Reflections
Problem U112 by Cezar Lupu and Valentin Vornicu

Let x, y, z be real numbers greater or equal to 1. Prove that

$$x^{x^3+2xyz}y^{y^3+2xyz}z^{z^3+2xyz} \geq \left(x^x y^y z^z\right)^{yz+zx+xy}.$$

Solution by Darij Grinberg.

Lemma 1. Let x, y, z, a, b, c be nonnegative reals such that $x \geq y \geq z$ and $ax \geq by$. Then,

$$\left(x^3+2xyz\right)a+\left(y^3+2xyz\right)b+\left(z^3+2xyz\right)c \geq \left(yz+zx+xy\right)\left(xa+yb+zc\right).$$

Proof of Lemma 1. Since $x \geq y \geq z$ and $ax \geq by$, the Vornicu-Schur inequality[1], applied to $A=x$, $B=y$, $C=z$, $X=ax$, $Y=by$, $Z=cz$, yields

$$ax\left(x-y\right)\left(x-z\right)+by\left(y-z\right)\left(y-x\right)+cz\left(z-x\right)\left(z-y\right) \geq 0.$$

This rewrites as

$$\left(x^3+2xyz\right)a+\left(y^3+2xyz\right)b+\left(z^3+2xyz\right)c-\left(yz+zx+xy\right)\left(xa+yb+zc\right) \geq 0.$$

Thus,

$$\left(x^3+2xyz\right)a+\left(y^3+2xyz\right)b+\left(z^3+2xyz\right)c \geq \left(yz+zx+xy\right)\left(xa+yb+zc\right),$$

proving Lemma 1.

Now let's solve the problem: Set $a=\ln x$, $b=\ln y$, $c=\ln z$. Then, a, b, c are nonnegative (since x, y, z are greater or equal to 1). WLOG assume that $x \geq y \geq z$ (we can assume this since the inequality is symmetric). Then, $ax \geq by$ (since a, b, x, y are nonnegative and $a \geq b$ and $x \geq y$, where $a \geq b$ is because $x \geq y$ yields $\underbrace{\ln x}_{=a} \geq \underbrace{\ln y}_{=b}$).

Hence, Lemma 1 yields

$$\left(x^3+2xyz\right)a+\left(y^3+2xyz\right)b+\left(z^3+2xyz\right)c \geq \left(yz+zx+xy\right)\left(xa+yb+zc\right).$$

Since

$$\begin{aligned}&\left(x^3+2xyz\right)a+\left(y^3+2xyz\right)b+\left(z^3+2xyz\right)c\\ &=\left(x^3+2xyz\right)\ln x+\left(y^3+2xyz\right)\ln y+\left(z^3+2xyz\right)\ln z=\ln\left(x^{x^3+2xyz}y^{y^3+2xyz}z^{z^3+2xyz}\right)\end{aligned}$$

and

$$\begin{aligned}\left(yz+zx+xy\right)\left(xa+yb+zc\right)&=\left(yz+zx+xy\right)\left(x\ln x+y\ln y+z\ln z\right)\\ &=\left(yz+zx+xy\right)\ln\left(x^x y^y z^z\right)=\ln\left(\left(x^x y^y z^z\right)^{yz+zx+xy}\right),\end{aligned}$$

this becomes $\ln\left(x^{x^3+2xyz}y^{y^3+2xyz}z^{z^3+2xyz}\right) \geq \ln\left(\left(x^x y^y z^z\right)^{yz+zx+xy}\right)$. Since the ln function is strictly increasing, this yields $x^{x^3+2xyz}y^{y^3+2xyz}z^{z^3+2xyz} \geq \left(x^x y^y z^z\right)^{yz+zx+xy}$, qed.

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[1] The "Vornicu-Schur inequality" that we use here is the following fact:

Let A, B, C be three reals, and let X, Y, Z be three nonnegative reals. If $A \geq B \geq C$ and $X \geq Y$, then

$$X\left(A-B\right)\left(A-C\right)+Y\left(B-C\right)\left(B-A\right)+Z\left(C-A\right)\left(C-B\right) \geq 0.$$

This is Theorem 1 **a)** in [1]. The proof is fairly easy (just show that $X\left(A-B\right)\left(A-C\right)+Y\left(B-C\right)\left(B-A\right) \geq 0$ and $Z\left(C-A\right)\left(C-B\right) \geq 0$).

References

[1] Darij Grinberg, *The Vornicu-Schur inequality and its variations*, version 13 August 2007.
`http://de.geocities.com/darij_grinberg/VornicuS.pdf`