Mathematical Reflections
Problem U111 by Titu Andreescu

Let n be a positive integer. For every $k \in \{0, 1, ..., n-1\}$, let $a_k = 2\cos\frac{\pi}{2^{n-k}}$. Prove that

$$\prod_{k=0}^{n-1}(1-a_k) = \frac{(-1)^{n-1}}{1+a_0}.$$

Solution by Darij Grinberg.

Lemma 1. For every $t \in \mathbb{R}$, we have

$$(2\cos t - 1)(2\cos t + 1) = 2\cos(2t) + 1.$$

Proof. We have

$$(2\cos t - 1)(2\cos t + 1) = 4\cos^2 t - 1 = 2\underbrace{\left(2\cos^2 t - 1\right)}_{=\cos(2t)} + 1 = 2\cos(2t) + 1,$$

and Lemma 1 is proven.

Lemma 2. For every $k \in \{0, 1, ..., n-1\}$, we have

$$a_k - 1 = \frac{a_{k+1}+1}{a_k+1},$$

where we set $a_n = -2$ (so that $a_k = 2\cos\frac{\pi}{2^{n-k}}$ holds for all $k \in \{0, 1, ..., n\}$).

Proof. We have $a_k + 1 \neq 0$ (since $a_k = 2\cos\underbrace{\frac{\pi}{2^{n-k}}}_{\in[0,\pi/2]} > 0$) and

$$\begin{aligned}(a_k - 1)(a_k + 1) &= \left(2\cos\frac{\pi}{2^{n-k}} - 1\right)\left(2\cos\frac{\pi}{2^{n-k}} + 1\right) = 2\cos\left(2\frac{\pi}{2^{n-k}}\right) + 1 \qquad \text{(by Lemma 1)}\\ &= 2\cos\frac{\pi}{2^{n-(k+1)}} + 1 = a_{k+1} + 1,\end{aligned}$$

so that $a_k - 1 = \frac{a_{k+1}+1}{a_k+1}$. Lemma 2 is proven.

Now,

$$\begin{aligned}\prod_{k=0}^{n-1}(1-a_k) &= \prod_{k=0}^{n-1}(-(a_k-1)) = (-1)^n\prod_{k=0}^{n-1}(a_k-1) = (-1)^n\prod_{k=0}^{n-1}\frac{a_{k+1}+1}{a_k+1} \qquad \text{(by Lemma 2)}\\ &= (-1)^n\frac{\prod_{k=0}^{n-1}(a_{k+1}+1)}{\prod_{k=0}^{n-1}(a_k+1)} = (-1)^n\frac{\prod_{k=1}^{n}(a_k+1)}{\prod_{k=0}^{n-1}(a_k+1)} = (-1)^n\frac{a_n+1}{a_0+1} = (-1)^n\frac{-2+1}{a_0+1}\\ &= (-1)^n\frac{-1}{a_0+1} = \frac{-(-1)^n}{1+a_0} = \frac{(-1)^{n-1}}{1+a_0}.\end{aligned}$$