Generalizations of Popoviciu's inequality

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Formal version; 4 March 2009

This is the "formal" version of my note *Generalizations of Popoviciu's inequality.* It contains the proofs with more details, but is much more burdensome to read because of this. I advise you to use this formal version only if you have troubles with understanding the standard version.

UPDATE: A glance into the survey [10], Chapter XVIII has revealed that most theorems in this paper are far from new. For instance, Theorem 5b was proven under weaker conditions (!) by Vasić and Stanković in [11]. Unfortunately, I have no access to [11] and the other references related to these inequalities.

Notation

First, we introduce a notation that we will use in the following paper: For any set S of numbers, we denote by $\max S$ the greatest element of the set S, and by $\min S$ the smallest element of S.

1. Introduction

In the last few years there was some activity on the MathLinks forum related to the Popoviciu inequality on convex functions. A number of generalizations were conjectured and subsequently proven using majorization theory and (mostly) a lot of computations. In this note I am presenting a probably new approach that proves these generalizations as well as some additional facts with a lesser amount of computation and avoiding the combinatorial difficulties of majorization theory (we will prove a version of the Karamata inequality on the way, but no prior knowledge of majorization theory is required - what we actually avoid is the asymmetric definition of majorization).

The very starting point of the whole theory is the following famous fact:

Theorem 1a, the Jensen inequality. Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Let x_1, x_2, ..., x_n be finitely many points from I. Then,

$$\frac{f(x_1) + f(x_2) + ... + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + ... + x_n}{n}\right).$$

In words, the arithmetic mean of the values of f at the points x_1, x_2, ..., x_n is greater or equal to the value of f at the arithmetic mean of these points.

We can obtain a "weighted version" of this inequality by replacing arithmetic means by weighted means with some nonnegative weights w_1, w_2, ..., w_n:

> **Theorem 1b, the weighted Jensen inequality.** Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Let x_1, x_2, ..., x_n be finitely many points from I. Let w_1, w_2, ..., w_n be n nonnegative reals which are not all equal to 0. Then,
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> $$\frac{w_1 f(x_1) + w_2 f(x_2) + ... + w_n f(x_n)}{w_1 + w_2 + ... + w_n} \geq f\left(\frac{w_1 x_1 + w_2 x_2 + ... + w_n x_n}{w_1 + w_2 + ... + w_n}\right).$$

Obviously, Theorem 1a follows from Theorem 1b applied to $w_1 = w_2 = ... = w_n = 1$, so that Theorem 1b is more general than Theorem 1a.

We won't stop at discussing equality cases here, since they can depend in various ways on the input (i. e., on the function f, the reals w_1, w_2, ..., w_n and the points x_1, x_2, ..., x_n) - but each time we use a result like Theorem 1b, with enough patience we can extract the equality case from the proof of this result and the properties of the input.

The Jensen inequality, in both of its versions above, is applied often enough to be called one of the main methods of proving inequalities. Now, in 1965, a similarly styled inequality was found by the Romanian Tiberiu Popoviciu:

> **Theorem 2a, the Popoviciu inequality.** Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$, and let x_1, x_2, x_3 be three points from I. Then,
>
> $$f(x_1)+f(x_2)+f(x_3)+3f\left(\frac{x_1+x_2+x_3}{3}\right) \geq 2f\left(\frac{x_2+x_3}{2}\right)+2f\left(\frac{x_3+x_1}{2}\right)+2f\left(\frac{x_1+x_2}{2}\right).$$

Again, a weighted version can be constructed:

> **Theorem 2b, the weighted Popoviciu inequality.** Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$, let x_1, x_2, x_3 be three points from I, and let w_1, w_2, w_3 be three nonnegative reals such that $w_2 + w_3 \neq 0$, $w_3 + w_1 \neq 0$ and $w_1 + w_2 \neq 0$. Then,
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> $$w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) + (w_1 + w_2 + w_3) f\left(\frac{w_1 x_1 + w_2 x_2 + w_3 x_3}{w_1 + w_2 + w_3}\right)$$
> $$\geq (w_2 + w_3) f\left(\frac{w_2 x_2 + w_3 x_3}{w_2 + w_3}\right) + (w_3 + w_1) f\left(\frac{w_3 x_3 + w_1 x_1}{w_3 + w_1}\right) + (w_1 + w_2) f\left(\frac{w_1 x_1 + w_2 x_2}{w_1 + w_2}\right).$$

Now, the really interesting part of the story began when Vasile Cîrtoaje - alias "Vasc" on the MathLinks forum - proposed the following two generalizations of Theorem 2a ([1] and [2] for Theorem 3a, and [1] and [3] for Theorem 4a):

> **Theorem 3a (Vasile Cîrtoaje).** Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Let x_1, x_2, ..., x_n be finitely many points from I. Then,
>
> $$\sum_{i=1}^{n} f(x_i)+n(n-2) f\left(\frac{x_1+x_2+...+x_n}{n}\right) \geq \sum_{j=1}^{n}(n-1) f\left(\frac{\sum\limits_{1\leq i\leq n;\ i\neq j} x_i}{n-1}\right).$$

Theorem 4a (Vasile Cîrtoaje). Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Let x_1, x_2, ..., x_n be finitely many points from I. Then,

$$(n-2)\sum_{i=1}^{n} f(x_i) + nf\left(\frac{x_1+x_2+...+x_n}{n}\right) \geq \sum_{1\leq i<j\leq n} 2f\left(\frac{x_i+x_j}{2}\right).$$

In [1], both of these facts were nicely proven by Cîrtoaje. I gave a different and rather long proof of Theorem 3a in [2]. All of these proofs use the Karamata inequality. Of course, Theorem 2a follows from each of the Theorems 3a and 4a upon setting $n = 3$.

It is pretty straightforward to obtain generalizations of Theorems 3a and 4a by putting in weights as in Theorems 1b and 2b. A more substantial generalization was given by Yufei Zhao - alias "Billzhao" on MathLinks - in [3]:

Theorem 5a (Yufei Zhao). Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Let x_1, x_2, ..., x_n be finitely many points from I, and let m be an integer. Then,

$$\binom{n-2}{m-1}\sum_{i=1}^{n} f(x_i) + \binom{n-2}{m-2} nf\left(\frac{x_1+x_2+...+x_n}{n}\right)$$
$$\geq \sum_{1\leq i_1<i_2<...<i_m\leq n} mf\left(\frac{x_{i_1}+x_{i_2}+...+x_{i_m}}{m}\right).$$

Note that if $m \leq 0$ or $m > n$, the sum $\sum\limits_{1\leq i_1<i_2<...<i_m\leq n} mf\left(\frac{x_{i_1}+x_{i_2}+...+x_{i_m}}{m}\right)$ is empty, so that its value is 0.

It is left to the reader to verify that Theorems 3a and 4a both are particular cases of Theorem 5a (in fact, set $m = n-1$ to get Theorem 3a and $m = 2$ to get Theorem 4a).

An elaborate proof of Theorem 5a was given by myself in [3]. After some time, the MathLinks user "Zhaobin" proposed a weighted version of this result:

Theorem 5b (Zhaobin). Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Let x_1, x_2, ..., x_n be finitely many points from I, let w_1, w_2, ..., w_n be nonnegative reals, and let m be an integer. Assume that $w_1 + w_2 + ... + w_n \neq 0$, and that $w_{i_1} + w_{i_2} + ... + w_{i_m} \neq 0$ for any m integers i_1, i_2, ..., i_m satisfying $1 \leq i_1 < i_2 < ... < i_m \leq n$.

Then,

$$\binom{n-2}{m-1}\sum_{i=1}^{n} w_i f(x_i) + \binom{n-2}{m-2}(w_1+w_2+...+w_n) f\left(\frac{w_1x_1+w_2x_2+...+w_nx_n}{w_1+w_2+...+w_n}\right)$$
$$\geq \sum_{1\leq i_1<i_2<...<i_m\leq n} (w_{i_1}+w_{i_2}+...+w_{i_m}) f\left(\frac{w_{i_1}x_{i_1}+w_{i_2}x_{i_2}+...+w_{i_m}x_{i_m}}{w_{i_1}+w_{i_2}+...+w_{i_m}}\right).$$

If we set $w_1 = w_2 = ... = w_n = 1$ in Theorem 5b, we obtain Theorem 5a. On the other hand, putting $n = 3$ and $m = 2$ in Theorem 5b, we get Theorem 2b.

In this note, I am going to prove Theorem 5b (and therefore also its particular cases - Theorems 2a, 2b, 3a, 4a and 5a). The proof is going to use no preknowledge - in particular, classical majorization theory will be avoided. Then, we are going to discuss an assertion analogous to Theorem 5b with its applications.

2. Absolute values interpolate convex functions

We start preparing for our proof by showing a classical property of convex functions[1]:

> **Theorem 6 (Hardy, Littlewood, Pólya).** Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Let x_1, x_2, ..., x_n be finitely many points from I. Then, there exist two real constants u and v and n nonnegative constants a_1, a_2, ..., a_n such that
>
> $$f(t) = vt + u + \sum_{i=1}^{n} a_i |t - x_i| \text{ holds for every } t \in \{x_1, x_2, ..., x_n\}.$$

In brief, this result states that every convex function $f(x)$ on n reals x_1, x_2, ..., x_n can be interpolated by a linear combination with nonnegative coefficients of a linear function and the n functions $|x - x_i|$.

The *proof of Theorem 6*, albeit technical, will be given here for the sake of completeness: First, we need a (very easy to prove) fact which I use to call the $\max\{0, x\}$ *formula:* For any real number x, we have $\max\{0, x\} = \frac{1}{2}(x + |x|)$.

Furthermore, we denote $f[y, z] = \frac{f(y) - f(z)}{y - z}$ for any two points y and z from I satisfying $y \neq z$. Then, we have $(y - z) \cdot f[y, z] = f(y) - f(z)$ for any two points y and z from I satisfying $y \neq z$.

We can assume that all points x_1, x_2, ..., x_n are pairwisely distinct (in fact, if we can find two different integers i_1 and i_2 from the set $\{1, 2, ..., n\}$ such that $x_{i_1} = x_{i_2}$, then we can just remove x_{i_2} from the list $(x_1, x_2, ..., x_n)$ and set $a_{i_2} = 0$, and since we have $\{x_1, x_2, ..., x_n\} = \{x_1, x_2, ..., x_{i_2-1}, x_{i_2+1}, ...x_n\}$, it remains to prove Theorem 6 for the $n - 1$ points x_1, x_2, ..., x_{i_2-1}, x_{i_2+1}, ..., x_n instead of all the n points x_1, x_2, ..., x_n; we can repeat this procedure as long as there are two equal points in the list $(x_1, x_2, ..., x_n)$, until we have reduced Theorem 6 to the case of a list of pairwisely distinct points). Therefore, we can WLOG assume that $x_1 < x_2 < ... < x_n$. Then, for

[1] This property appeared as Proposition B.4 in [8], which refers to [9] for its origins. It was also mentioned by a MathLinks user called "Fleeting_Guest" in [4], post #18 as a known fact, albeit in a slightly different (but equivalent) form.

every $j \in \{1, 2, ..., n\}$, we have

$$\begin{aligned}
f(x_j) &= f(x_1) + \sum_{k=1}^{j-1} (f(x_{k+1}) - f(x_k)) = f(x_1) + \sum_{k=1}^{j-1} (x_{k+1} - x_k) \cdot f[x_{k+1}, x_k] \\
&= f(x_1) + \sum_{k=1}^{j-1} (x_{k+1} - x_k) \cdot \left(f[x_2, x_1] + \sum_{i=2}^{k} (f[x_{i+1}, x_i] - f[x_i, x_{i-1}]) \right) \\
&= f(x_1) + \sum_{k=1}^{j-1} (x_{k+1} - x_k) \cdot f[x_2, x_1] + \sum_{k=1}^{j-1} (x_{k+1} - x_k) \cdot \sum_{i=2}^{k} (f[x_{i+1}, x_i] - f[x_i, x_{i-1}]) \\
&= f(x_1) + f[x_2, x_1] \cdot \sum_{k=1}^{j-1} (x_{k+1} - x_k) + \sum_{k=1}^{j-1} \sum_{i=2}^{k} (f[x_{i+1}, x_i] - f[x_i, x_{i-1}]) \cdot (x_{k+1} - x_k) \\
&= f(x_1) + f[x_2, x_1] \cdot \sum_{k=1}^{j-1} (x_{k+1} - x_k) + \sum_{i=2}^{j-1} \sum_{k=i}^{j-1} (f[x_{i+1}, x_i] - f[x_i, x_{i-1}]) \cdot (x_{k+1} - x_k) \\
&= f(x_1) + f[x_2, x_1] \cdot \sum_{k=1}^{j-1} (x_{k+1} - x_k) + \sum_{i=2}^{j-1} (f[x_{i+1}, x_i] - f[x_i, x_{i-1}]) \cdot \sum_{k=i}^{j-1} (x_{k+1} - x_k) \\
&= f(x_1) + f[x_2, x_1] \cdot (x_j - x_1) + \sum_{i=2}^{j-1} (f[x_{i+1}, x_i] - f[x_i, x_{i-1}]) \cdot (x_j - x_i).
\end{aligned}$$

Now we set

$$\begin{aligned}
&\alpha_1 = \alpha_n = 0; \\
&\alpha_i = f[x_{i+1}, x_i] - f[x_i, x_{i-1}] \qquad \text{for all } i \in \{2, 3, ..., n-1\}.
\end{aligned}$$

Using these notations, the above computation becomes

$$\begin{aligned}
f\left(x_j\right) &= f\left(x_1\right) + f\left[x_2, x_1\right] \cdot \left(x_j - x_1\right) + \sum_{i=2}^{j-1} \alpha_i \cdot \left(x_j - x_i\right) \\
&= f\left(x_1\right) + f\left[x_2, x_1\right] \cdot \left(x_j - x_1\right) + \underbrace{0}_{=\alpha_1} \cdot \max\left\{0, x_j - x_1\right\} \\
&+ \sum_{i=2}^{j-1} \alpha_i \cdot \underbrace{\left(x_j - x_i\right)}_{=\max\left\{0, x_j - x_i\right\}, \text{ since } x_j - x_i \geq 0, \text{ as } x_i \leq x_j} + \sum_{i=j}^{n} \alpha_i \cdot \underbrace{0}_{=\max\left\{0, x_j - x_i\right\}, \text{ since } x_j - x_i \leq 0, \text{ as } x_j \leq x_i} \\
&= f\left(x_1\right) + f\left[x_2, x_1\right] \cdot \left(x_j - x_1\right) + \alpha_1 \cdot \max\left\{0, x_j - x_1\right\} \\
&+ \sum_{i=2}^{j-1} \alpha_i \cdot \max\left\{0, x_j - x_i\right\} + \sum_{i=j}^{n} \alpha_i \cdot \max\left\{0, x_j - x_i\right\} \\
&= f\left(x_1\right) + f\left[x_2, x_1\right] \cdot \left(x_j - x_1\right) + \sum_{i=1}^{n} \alpha_i \cdot \max\left\{0, x_j - x_i\right\} \\
&= f\left(x_1\right) + f\left[x_2, x_1\right] \cdot \left(x_j - x_1\right) + \sum_{i=1}^{n} \alpha_i \cdot \frac{1}{2}\left(\left(x_j - x_i\right) + \left|x_j - x_i\right|\right) \\
&\qquad \left(\text{since } \max\left\{0, x_j - x_i\right\} = \frac{1}{2}\left(\left(x_j - x_i\right) + \left|x_j - x_i\right|\right) \text{ by the } \max\left\{0, x\right\} \text{ formula}\right) \\
&= f\left(x_1\right) + f\left[x_2, x_1\right] \cdot \left(x_j - x_1\right) + \sum_{i=1}^{n} \alpha_i \cdot \frac{1}{2}\left(x_j - x_i\right) + \sum_{i=1}^{n} \alpha_i \cdot \frac{1}{2}\left|x_j - x_i\right| \\
&= f\left(x_1\right) + \left(f\left[x_2, x_1\right] x_j - f\left[x_2, x_1\right] x_1\right) + \left(\frac{1}{2}\sum_{i=1}^{n} \alpha_i x_j - \frac{1}{2}\sum_{i=1}^{n} \alpha_i x_i\right) + \sum_{i=1}^{n} \frac{1}{2}\alpha_i \left|x_j - x_i\right| \\
&= \left(f\left[x_2, x_1\right] + \frac{1}{2}\sum_{i=1}^{n} \alpha_i\right) x_j + \left(f\left(x_1\right) - f\left[x_2, x_1\right] x_1 - \frac{1}{2}\sum_{i=1}^{n} \alpha_i x_i\right) + \sum_{i=1}^{n} \frac{1}{2}\alpha_i \left|x_j - x_i\right|.
\end{aligned}$$

Thus, if we denote

$$v = f\left[x_2, x_1\right] + \frac{1}{2}\sum_{i=1}^{n} \alpha_i; \qquad u = f\left(x_1\right) - f\left[x_2, x_1\right] x_1 - \frac{1}{2}\sum_{i=1}^{n} \alpha_i x_i;$$

$$a_i = \frac{1}{2}\alpha_i \qquad \text{for all } i \in \left\{1, 2, ..., n\right\},$$

then we have

$$f\left(x_j\right) = v x_j + u + \sum_{i=1}^{n} a_i \left|x_j - x_i\right|.$$

Since we have shown this for every $j \in \left\{1, 2, ..., n\right\}$, we can restate this as follows: We have

$$f\left(t\right) = vt + u + \sum_{i=1}^{n} a_i \left|t - x_i\right| \text{ for every } t \in \left\{x_1, x_2, ..., x_n\right\}.$$

Hence, in order for the proof of Theorem 6 to be complete, it is enough to show that the n reals a_1, a_2, ..., a_n are nonnegative. Since $a_i = \frac{1}{2}\alpha_i$ for every $i \in \left\{1, 2, ..., n\right\}$, this will follow once it is proven that the n reals α_1, α_2, ..., α_n are nonnegative. Thus, we have to show that α_i is nonnegative for every $i \in \{1, 2, ..., n\}$. This is trivial for $i = 1$ and for $i = n$ (since $\alpha_1 = 0$ and $\alpha_n = 0$), so it remains to prove that α_i is nonnegative for every $i \in \{2, 3, ..., n-1\}$. Now, since $\alpha_i = f[x_{i+1}, x_i] - f[x_i, x_{i-1}]$ for every $i \in \{2, 3, ..., n-1\}$, we thus have to show that $f[x_{i+1}, x_i] - f[x_i, x_{i-1}]$ is nonnegative for every $i \in \{2, 3, ..., n-1\}$. In other words, we have to prove that $f[x_{i+1}, x_i] \geq f[x_i, x_{i-1}]$ for every $i \in \{2, 3, ..., n-1\}$. But since $x_{i-1} < x_i < x_{i+1}$, this follows from the next lemma:

Lemma 7. Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Let x, y, z be three points from I satisfying $x < y < z$. Then, $f[z, y] \geq f[y, x]$.

Proof of Lemma 7. Since the function f is convex on I, and since z and x are points from I, the definition of convexity yields

$$\frac{\frac{1}{z-y} f(z) + \frac{1}{y-x} f(x)}{\frac{1}{z-y} + \frac{1}{y-x}} \geq f\left(\frac{\frac{1}{z-y} z + \frac{1}{y-x} x}{\frac{1}{z-y} + \frac{1}{y-x}}\right)$$

(here we have used that $\frac{1}{z-y} > 0$ and $\frac{1}{y-x} > 0$, what is clear from $x < y < z$). Since $\frac{\frac{1}{z-y} z + \frac{1}{y-x} x}{\frac{1}{z-y} + \frac{1}{y-x}} = y$, this simplifies to

$$\frac{\frac{1}{z-y} f(z) + \frac{1}{y-x} f(x)}{\frac{1}{z-y} + \frac{1}{y-x}} \geq f(y), \qquad \text{so that}$$

$$\frac{1}{z-y} f(z) + \frac{1}{y-x} f(x) \geq \left(\frac{1}{z-y} + \frac{1}{y-x}\right) f(y), \qquad \text{so that}$$

$$\frac{1}{z-y} f(z) + \frac{1}{y-x} f(x) \geq \frac{1}{z-y} f(y) + \frac{1}{y-x} f(y), \qquad \text{so that}$$

$$\frac{1}{z-y} f(z) - \frac{1}{z-y} f(y) \geq \frac{1}{y-x} f(y) - \frac{1}{y-x} f(x), \qquad \text{so that}$$

$$\frac{f(z) - f(y)}{z-y} \geq \frac{f(y) - f(x)}{y-x}.$$

This becomes $f[z, y] \geq f[y, x]$, and thus Lemma 7 is proven. Thus, the proof of Theorem 6 is completed.

3. The Karamata inequality in symmetric form

Now as Theorem 6 is proven, it becomes easy to prove the Karamata inequality in the following form:

Theorem 8a, the Karamata inequality in symmetric form. Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$, and let n be a positive integer. Let x_1, x_2, ..., x_n, y_1, y_2, ..., y_n be $2n$ points from I. Assume that

$$|x_1 - t| + |x_2 - t| + ... + |x_n - t| \geq |y_1 - t| + |y_2 - t| + ... + |y_n - t|$$

holds for every $t \in \{x_1, x_2, ..., x_n, y_1, y_2, ..., y_n\}$. Then,

$$f(x_1) + f(x_2) + ... + f(x_n) \geq f(y_1) + f(y_2) + ... + f(y_n).$$

This is a particular case of the following result:

Theorem 8b, the weighted Karamata inequality in symmetric form. Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$, and let N be a positive integer. Let z_1, z_2, ..., z_N be N points from I, and let w_1, w_2, ..., w_N be N reals. Assume that

$$\sum_{k=1}^{N} w_k = 0, \qquad (1)$$

and that

$$\sum_{k=1}^{N} w_k |z_k - t| \geq 0 \text{ holds for every } t \in \{z_1, z_2, ..., z_N\}. \qquad (2)$$

Then,

$$\sum_{k=1}^{N} w_k f(z_k) \geq 0. \qquad (3)$$

It is very easy to conclude Theorem 8a from Theorem 8b; we postpone this argument until Theorem 8b is proven.

Time for *a remark to readers familiar with majorization theory.* One may wonder why I call the two results above "Karamata inequalities". In fact, the Karamata inequality in its most known form claims:

Theorem 9, the Karamata inequality. Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$, and let n be a positive integer. Let x_1, x_2, ..., x_n, y_1, y_2, ..., y_n be $2n$ points from I such that $(x_1, x_2, ..., x_n) \succ (y_1, y_2, ..., y_n)$. Then,

$$f(x_1) + f(x_2) + ... + f(x_n) \geq f(y_1) + f(y_2) + ... + f(y_n).$$

According to [2], post #11, Lemma 1, the condition $(x_1, x_2, ..., x_n) \succ (y_1, y_2, ..., y_n)$ yields that $|x_1 - t| + |x_2 - t| + ... + |x_n - t| \geq |y_1 - t| + |y_2 - t| + ... + |y_n - t|$ holds for every real t - and thus, in particular, for every $t \in \{z_1, z_2, ..., z_n\}$. Hence, whenever the condition of Theorem 9 holds, the condition of Theorem 8a holds as well. Thus, Theorem 9 follows from Theorem 8a. With just a little more work, we could also derive Theorem 8a from Theorem 9, so that Theorems 8a and 9 are equivalent.

Note that Theorem 8b is more general than the Fuchs inequality (a more well-known weighted version of the Karamata inequality). See [5] for a generalization of majorization theory to weighted families of points (apparently already known long time ago), with a different approach to this fact.

As promised, here is a *proof of Theorem 8b:* First, substituting $t = \max\{z_1, z_2, ..., z_N\}$ into (2) (it is clear that this t satisfies $t \in \{z_1, z_2, ..., z_N\}$), we get $\sum\limits_{k=1}^{N} w_k |z_k - t| \geq 0$, what is equivalent to $-\sum\limits_{k=1}^{N} w_k z_k \geq 0$ (since $t = \max\{z_1, z_2, ..., z_N\}$ yields $z_k \leq t$ for every $k \in \{1, 2, ..., N\}$, so that $z_k - t \leq 0$ and thus $|z_k - t| = -(z_k - t) = t - z_k$ for every $k \in \{1, 2, ..., N\}$, so that

$$\sum_{k=1}^{N} w_k |z_k - t| = \sum_{k=1}^{N} w_k (t - z_k) = t \underbrace{\sum_{k=1}^{N} w_k}_{=0} - \sum_{k=1}^{N} w_k z_k = t \cdot 0 - \sum_{k=1}^{N} w_k z_k = -\sum_{k=1}^{N} w_k z_k$$

). Hence, $\sum\limits_{k=1}^{N} w_k z_k \leq 0$.

On the other hand, substituting $t = \min\{z_1, z_2, ..., z_N\}$ into (2) (again, it is clear that this t satisfies $t \in \{z_1, z_2, ..., z_N\}$), we get $\sum\limits_{k=1}^{N} w_k |z_k - t| \geq 0$, what is equivalent to $\sum\limits_{k=1}^{N} w_k z_k \geq 0$ (since $t = \min\{z_1, z_2, ..., z_N\}$ yields $z_k \geq t$ for every $k \in \{1, 2, ..., N\}$, so that $z_k - t \geq 0$ and thus $|z_k - t| = z_k - t$ for every $k \in \{1, 2, ..., N\}$, so that

$$\sum_{k=1}^{N} w_k |z_k - t| = \sum_{k=1}^{N} w_k (z_k - t) = \sum_{k=1}^{N} w_k z_k - t \underbrace{\sum_{k=1}^{N} w_k}_{=0} = \sum_{k=1}^{N} w_k z_k - t \cdot 0 = \sum_{k=1}^{N} w_k z_k$$

).

Combining $\sum\limits_{k=1}^{N} w_k z_k \leq 0$ with $\sum\limits_{k=1}^{N} w_k z_k \geq 0$, we get $\sum\limits_{k=1}^{N} w_k z_k = 0$.

The function $f : I \to \mathbb{R}$ is convex, and z_1, z_2, ..., z_N are finitely many points from I. Hence, Theorem 6 yields the existence of two real constants u and v and N nonnegative constants a_1, a_2, ..., a_N such that

$$f(t) = vt + u + \sum_{i=1}^{N} a_i |t - z_i| \text{ holds for every } t \in \{z_1, z_2, ..., z_N\}.$$

Thus,

$$f(z_k) = vz_k + u + \sum_{i=1}^{N} a_i |z_k - z_i| \qquad \text{for every } k \in \{1, 2, ..., N\}$$

(since $z_k \in \{z_1, z_2, ..., z_N\}$). Hence,

$$\sum_{k=1}^{N} w_k f(z_k) = \sum_{k=1}^{N} w_k \left(v z_k + u + \sum_{i=1}^{N} a_i |z_k - z_i| \right) = v \underbrace{\sum_{k=1}^{N} w_k z_k}_{=0} + u \underbrace{\sum_{k=1}^{N} w_k}_{=0} + \sum_{k=1}^{N} w_k \sum_{i=1}^{N} a_i |z_k - z_i|$$

$$= \sum_{k=1}^{N} w_k \sum_{i=1}^{N} a_i |z_k - z_i| = \sum_{i=1}^{N} a_i \underbrace{\sum_{k=1}^{N} w_k |z_k - z_i|}_{\geq 0 \text{ according to (2) for } t = z_i} \geq 0.$$

Thus, Theorem 8b is proven.

Now, as Theorem 8b is verified, let us conclude Theorem 8a:

Proof of Theorem 8a: Set $N = 2n$ and

$$z_k = \begin{cases} x_k \text{ for all } k \in \{1, 2, ..., n\}; \\ y_{k-n} \text{ for all } k \in \{n+1, n+2, ..., 2n\} \end{cases}; \qquad w_k = \begin{cases} 1 \text{ for all } k \in \{1, 2, ..., n\}; \\ -1 \text{ for all } k \in \{n+1, n+2, ..., 2n\} \end{cases}.$$

That is,

$$z_1 = x_1, \qquad z_2 = x_2, \qquad ..., \qquad z_n = x_n;$$
$$z_{n+1} = y_1, \qquad z_{n+2} = y_2, \qquad ..., \qquad z_{2n} = y_n;$$
$$w_1 = w_2 = ... = w_n = 1; \qquad w_{n+1} = w_{n+2} = ... = w_{2n} = -1.$$

Then, the conditions of Theorem 8b are fulfilled: In fact, (1) is fulfilled because

$$\sum_{k=1}^{N} w_k = \sum_{k=1}^{2n} w_k = \sum_{k=1}^{n} w_k + \sum_{k=n+1}^{2n} w_k = \sum_{k=1}^{n} 1 + \sum_{k=n+1}^{2n} (-1) = n \cdot 1 + n \cdot (-1) = 0.$$

Also, (2) is fulfilled, because for every $t \in \{z_1, z_2, ..., z_N\}$, we have $t \in \{x_1, x_2, ..., x_n, y_1, y_2, ..., y_n\}$ (because $\{z_1, z_2, ..., z_N\} = \{x_1, x_2, ..., x_n, y_1, y_2, ..., y_n\}$) and thus, after the condition of Theorem 8a, we have

$$|x_1 - t| + |x_2 - t| + ... + |x_n - t| \geq |y_1 - t| + |y_2 - t| + ... + |y_n - t|, \qquad \text{so that}$$

$$\sum_{k=1}^{n} |x_k - t| \geq \sum_{k=1}^{n} |y_k - t|, \qquad \text{so that}$$

$$\sum_{k=1}^{n} |x_k - t| - \sum_{k=1}^{n} |y_k - t| \geq 0,$$

and thus

$$\sum_{k=1}^{N} w_k |z_k - t| = \sum_{k=1}^{2n} w_k |z_k - t| = \sum_{k=1}^{n} w_k |z_k - t| + \sum_{k=n+1}^{2n} w_k |z_k - t|$$
$$= \sum_{k=1}^{n} 1 \cdot |x_k - t| + \sum_{k=n+1}^{2n} (-1) \cdot |y_{k-n} - t|$$
$$= \sum_{k=1}^{n} |x_k - t| - \sum_{k=n+1}^{2n} |y_{k-n} - t| = \sum_{k=1}^{n} |x_k - t| - \sum_{k=1}^{n} |y_k - t| \geq 0,$$

what proves (2).

Hence, we can apply Theorem 8b and obtain

$$\sum_{k=1}^{N} w_k f(z_k) \geq 0.$$

That is,

$$0 \leq \sum_{k=1}^{N} w_k f(z_k) = \sum_{k=1}^{2n} w_k f(z_k) = \sum_{k=1}^{n} w_k f(z_k) + \sum_{k=n+1}^{2n} w_k f(z_k) = \sum_{k=1}^{n} 1 f(x_k) + \sum_{k=n+1}^{2n} (-1) f(y_{k-n})$$
$$= \sum_{k=1}^{n} f(x_k) - \sum_{k=n+1}^{2n} f(y_{k-n}) = \sum_{k=1}^{n} f(x_k) - \sum_{k=1}^{n} f(y_k),$$

so that $\sum\limits_{k=1}^{n} f(x_k) \geq \sum\limits_{k=1}^{n} f(y_k)$, and Theorem 8a is proven.

4. A property of zero-sum vectors

Next, we are going to show some properties of real vectors.

If k is an integer and $v \in \mathbb{R}^k$ is a vector, then, for any $i \in \{1, 2, ..., k\}$, we denote by v_i the i-th coordinate of the vector v. Then, $v = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_k \end{pmatrix}$.

Let n be a positive integer. We consider the vector space $\mathbb{R}^n$. Let $(e_1, e_2, ..., e_n)$ be the standard basis of this vector space $\mathbb{R}^n$; in other words, for every $i \in \{1, 2, ..., n\}$, let e_i be the vector from $\mathbb{R}^n$ such that $(e_i)_i = 1$ and $(e_i)_j = 0$ for every $j \in \{1, 2, ..., n\} \setminus \{i\}$. Let V_n be the subspace of $\mathbb{R}^n$ defined by

$$V_n = \{x \in \mathbb{R}^n \mid x_1 + x_2 + ... + x_n = 0\}.$$

For any $u \in \{1, 2, ..., n\}$ and any two *distinct* numbers i and j from the set $\{1, 2, ..., n\}$, we have

$$(e_i - e_j)_u = \begin{cases} 1, & \text{if } u = i; \\ -1, & \text{if } u = j; \\ 0, & \text{if } u \neq i \text{ and } u \neq j \end{cases}. \tag{4}$$

We have $e_i - e_j \in V_n$ for any two numbers i and j from the set $\{1, 2, ..., n\}$ (in fact, if the numbers i and j are distinct, then (4) yields $(e_i - e_j)_1 + (e_i - e_j)_2 + ... + (e_i - e_j)_n = 0$ and thus $e_i - e_j \in V_n$, and if not, then $i = j$ and thus $e_i - e_j = e_j - e_j = 0 \in V_n$).

For any vector $t \in \mathbb{R}^n$, we denote $I(t) = \{k \in \{1, 2, ..., n\} \mid t_k > 0\}$ and $J(t) = \{k \in \{1, 2, ..., n\} \mid t_k < 0\}$. Obviously, for every $t \in \mathbb{R}^n$, the sets $I(t)$ and $J(t)$ are disjoint (since there does not exist any k satisfying both $t_k > 0$ and $t_k < 0$).

Now we are going to show:

Theorem 10. Let n be a positive integer. Let $x \in V_n$ be a vector. Then, there exist nonnegative reals $a_{i,j}$ for all pairs $(i, j) \in I(x) \times J(x)$ such that

$$x = \sum_{(i,j) \in I(x) \times J(x)} a_{i,j} (e_i - e_j).$$

Proof of Theorem 10. We will prove Theorem 10 by induction over $|I(x)| + |J(x)|$.

The *basis of the induction* - the case when $|I(x)| + |J(x)| = 0$ - is trivial: If $|I(x)| + |J(x)| = 0$, then $I(x) = J(x) = \varnothing$, so that $x = \sum\limits_{(i,j) \in I(x) \times J(x)} a_{i,j}(e_i - e_j)$ holds because $x = 0$ (because if x were different from 0, then there would exist at least one $k \in \{1, 2, ..., n\}$ such that $x_k \neq 0$, so that either $x_k > 0$ or $x_k < 0$, but $x_k > 0$ is impossible because $\{k \in \{1, 2, ..., n\} \mid x_k > 0\} = I(x) = \varnothing$, and $x_k < 0$ is impossible because $\{k \in \{1, 2, ..., n\} \mid x_k < 0\} = J(x) = \varnothing$) and $\sum\limits_{(i,j) \in I(x) \times J(x)} a_{i,j}(e_i - e_j) = 0$ (since $I(x) = J(x) = \varnothing$ yields $I(x) \times J(x) = \varnothing$, so that $\sum\limits_{(i,j) \in I(x) \times J(x)} a_{i,j}(e_i - e_j)$ is an empty sum and thus equals 0).

Now we come to the *induction step*: Let r be a positive integer. Assume that Theorem 10 holds for all $x \in V_n$ with $|I(x)| + |J(x)| < r$. We have to show that Theorem 10 holds for all $x \in V_n$ with $|I(x)| + |J(x)| = r$.

In order to prove this, we let $z \in V_n$ be an arbitrary vector with $|I(z)| + |J(z)| = r$. We then have to prove that Theorem 10 holds for $x = z$. In other words, we have to show that there exist nonnegative reals $a_{i,j}$ for all pairs $(i, j) \in I(z) \times J(z)$ such that

$$z = \sum_{(i,j) \in I(z) \times J(z)} a_{i,j}(e_i - e_j). \tag{5}$$

First, $|I(z)| + |J(z)| = r$ and $r > 0$ yield $|I(z)| + |J(z)| > 0$. Hence, at least one of the sets $I(z)$ and $J(z)$ is non-empty.

Now, since $z \in V_n$, we have $z_1 + z_2 + ... + z_n = 0$. Hence, either $z_k = 0$ for every $k \in \{1, 2, ..., n\}$, or there is at least one positive number and at least one negative number in the set $\{z_1, z_2, ..., z_n\}$. The first case is impossible (in fact, if $z_k = 0$ for every $k \in \{1, 2, ..., n\}$, then $I(z) = \{k \in \{1, 2, ..., n\} \mid z_k > 0\} = \varnothing$ and similarly $J(z) = \varnothing$, contradicting the fact that at least one of the sets $I(z)$ and $J(z)$ is non-empty). Thus, the second case must hold - i. e., there is at least one positive number and at least one negative number in the set $\{z_1, z_2, ..., z_n\}$. In other words, there exists a number $u \in \{1, 2, ..., n\}$ such that $z_u > 0$, and a number $v \in \{1, 2, ..., n\}$ such that $z_v < 0$. Of course, $z_u > 0$ yields $u \in I(z)$, and $z_v < 0$ yields $v \in J(z)$. Needless to say that $u \neq v$ (since $z_u > 0$ and $z_v < 0$).

Now, we distinguish between two cases: the *first case* will be the case when $z_u + z_v \geq 0$, and the *second case* will be the case when $z_u + z_v \leq 0$.

Let us consider the *first case:* In this case, $z_u + z_v \geq 0$. Then, let $z' = z + z_v(e_u - e_v)$. Since $z \in V_n$ and $e_u - e_v \in V_n$, we have $z + z_v(e_u - e_v) \in V_n$ (since V_n is a vector space), so that $z' \in V_n$. From $z' = z + z_v(e_u - e_v)$, the coordinate representation of the vector z' is easily obtained:

$$z' = \begin{pmatrix} z'_1 \\ z'_2 \\ ... \\ z'_n \end{pmatrix}, \qquad \text{where } \begin{cases} z'_k = z_k \text{ for all } k \in \{1, 2, ..., n\} \setminus \{u, v\}; \\ z'_u = z_u + z_v; \\ z'_v = 0 \end{cases}.$$

Thus,

$$\begin{aligned}
I(z') &= \{k \in \{1,2,...,n\} \mid z'_k > 0\} \\
&= \{k \in \{1,2,...,n\} \setminus \{u,v\} \mid z'_k > 0\} \cup \{k = u \mid z'_k > 0\} \cup \underbrace{\{k = v \mid z'_k > 0\}}_{=\varnothing,\text{ since } z'_v \text{ is not } >0,\text{ but } =0} \\
&= \{k \in \{1,2,...,n\} \setminus \{u,v\} \mid z'_k > 0\} \cup \{k = u \mid z'_k > 0\} \\
&= \underbrace{\{k \in \{1,2,...,n\} \setminus \{u,v\} \mid z_k > 0\}}_{\text{subset of } \{k\in\{1,2,...,n\} \mid z_k>0\}=I(z)} \cup \underbrace{\{k = u \mid z'_k > 0\}}_{\text{this is either } \{u\} \text{ or } \varnothing,\text{ anyway a subset of } I(z) \text{ since } u\in I(z)} \\
&\qquad (\text{we have replaced } z'_k \text{ by } z_k \text{ here, since } z'_k = z_k \text{ for all } k \in \{1,2,...,n\} \setminus \{u,v\}) \\
&\subseteq I(z)
\end{aligned}$$

(since the union of two subsets of $I(z)$ must be a subset of $I(z)$). Thus, $|I(z')| \leq |I(z)|$. Besides, $z'_u \geq 0$ (since $z'_u = z_u + z_v \geq 0$), so that

$$\begin{aligned}
J(z') &= \{k \in \{1,2,...,n\} \mid z'_k < 0\} \\
&= \{k \in \{1,2,...,n\} \setminus \{u,v\} \mid z'_k < 0\} \cup \underbrace{\{k = u \mid z'_k < 0\}}_{=\varnothing,\text{ since } z'_u \text{ is not } <0,\text{ but } \geq 0} \cup \underbrace{\{k = v \mid z'_k < 0\}}_{=\varnothing,\text{ since } z'_v \text{ is not } <0,\text{ but } =0} \\
&= \{k \in \{1,2,...,n\} \setminus \{u,v\} \mid z'_k < 0\} \\
&= \{k \in \{1,2,...,n\} \setminus \{u,v\} \mid z_k < 0\} \\
&\qquad (\text{we have replaced } z'_k \text{ by } z_k \text{ here, since } z'_k = z_k \text{ for all } k \in \{1,2,...,n\} \setminus \{u,v\}) \\
&\subseteq \{k \in \{1,2,...,n\} \mid z_k < 0\} = J(z).
\end{aligned}$$

Moreover, $J(z')$ is a proper subset of $J(z)$, because $v \notin J(z')$ (since z'_v is not < 0, but $= 0$) but $v \in J(z)$. Hence, $|J(z')| < |J(z)|$. Combined with $|I(z')| \leq |I(z)|$, this yields $|I(z')| + |J(z')| < |I(z)| + |J(z)|$. In view of $|I(z)| + |J(z)| = r$, this becomes $|I(z')| + |J(z')| < r$. Thus, since we have assumed that Theorem 10 holds for all $x \in V_n$ with $|I(x)| + |J(x)| < r$, we can apply Theorem 10 to $x = z'$, and we see that there exist nonnegative reals $a'_{i,j}$ for all pairs $(i,j) \in I(z') \times J(z')$ such that

$$z' = \sum_{(i,j) \in I(z') \times J(z')} a'_{i,j} (e_i - e_j).$$

Now, $z' = z + z_v (e_u - e_v)$ yields $z = z' - z_v (e_u - e_v)$. Since $z_v < 0$, we have $-z_v > 0$, so that, particularly, $-z_v$ is nonnegative.

Since $I(z') \subseteq I(z)$ and $J(z') \subseteq J(z)$, we have $I(z') \times J(z') \subseteq I(z) \times J(z)$. Also, $(u,v) \in I(z) \times J(z)$ (because $u \in I(z)$ and $v \in J(z)$) and $(u,v) \notin I(z') \times J(z')$ (because $v \notin J(z')$).

Hence, the sets $I(z') \times J(z')$ and $\{(u,v)\}$ are two disjoint subsets of the set $I(z) \times J(z)$. We can thus define nonnegative reals $a_{i,j}$ for all pairs $(i,j) \in I(z) \times J(z)$ by setting

$$a_{i,j} = \begin{cases} a'_{i,j}, & \text{if } (i,j) \in I(z') \times J(z'); \\ -z_v, & \text{if } (i,j) = (u,v); \\ 0, & \text{if neither of the two cases above holds} \end{cases}$$

(these $a_{i,j}$ are all nonnegative because $a'_{i,j}$, $-z_v$ and 0 are nonnegative). Then,

$$\begin{aligned}
&\sum_{(i,j)\in I(z)\times J(z)} a_{i,j}\left(e_i-e_j\right)\\
&=\sum_{(i,j)\in I(z')\times J(z')} a_{i,j}\left(e_i-e_j\right)+\sum_{(i,j)=(u,v)} a_{i,j}\left(e_i-e_j\right)+\sum_{(i,j)\in(I(z)\times J(z))\setminus((I(z')\times J(z'))\cup\{(u,v)\})} a_{i,j}\left(e_i-e_j\right)\\
&=\sum_{(i,j)\in I(z')\times J(z')} a'_{i,j}\left(e_i-e_j\right)+\sum_{(i,j)=(u,v)} \left(-z_v\right)\left(e_i-e_j\right)+\sum_{(i,j)\in(I(z)\times J(z))\setminus((I(z')\times J(z'))\cup\{(u,v)\})} 0\left(e_i-e_j\right)\\
&=\sum_{(i,j)\in I(z')\times J(z')} a'_{i,j}\left(e_i-e_j\right)+\left(-z_v\right)\left(e_u-e_v\right)+0=z'+\left(-z_v\right)\left(e_u-e_v\right)+0\\
&=\left(z+z_v\left(e_u-e_v\right)\right)+\left(-z_v\right)\left(e_u-e_v\right)+0=z.
\end{aligned}$$

Thus, (5) is fulfilled.

Similarly, we can fulfill (5) in the *second case*, repeating the arguments we have done for the first case while occasionally interchanging u with v, as well as I with J, as well as $<$ with $>$. Here is a brief outline of how we have to proceed in the second case: Denote $z'=z-z_u\left(e_u-e_v\right)$. Show that $z'\in V_n$ (as in the first case). Notice that

$$z'=\begin{pmatrix} z'_1\\ z'_2\\ \ldots\\ z'_n\end{pmatrix},\qquad \text{where }\begin{cases} z'_k=z_k \text{ for all } k\in\{1,2,\ldots,n\}\setminus\{u,v\};\\ z'_u=0;\\ z'_v=z_u+z_v\end{cases}.$$

Prove that $u\notin I(z')$ (as we proved $v\notin J(z')$ in the first case). Prove that $J(z')\subseteq J(z)$ (similarly to the proof of $I(z')\subseteq I(z)$ in the first case) and that $I(z')$ is a proper subset of $I(z)$ (similarly to the proof that $J(z')$ is a proper subset of $J(z)$ in the first case). Show that there exist nonnegative reals $a'_{i,j}$ for all pairs $(i,j)\in I(z')\times J(z')$ such that

$$z'=\sum_{(i,j)\in I(z')\times J(z')} a'_{i,j}\left(e_i-e_j\right)$$

(as in the first case). Note that z_u is nonnegative (since $z_u>0$). Prove that the sets $I(z')\times J(z')$ and $\{(u,v)\}$ are two disjoint subsets of the set $I(z)\times J(z)$ (as in the first case). Define nonnegative reals $a_{i,j}$ for all pairs $(i,j)\in I(z)\times J(z)$ by setting

$$a_{i,j}=\begin{cases} a'_{i,j}, \text{ if } (i,j)\in I(z')\times J(z');\\ z_u, \text{ if } (i,j)=(u,v);\\ 0, \text{ if neither of the two cases above holds}\end{cases}.$$

Prove that these nonnegative reals $a_{i,j}$ fulfill (5).

Thus, in each of the two cases, we have proven that there exist nonnegative reals $a_{i,j}$ for all pairs $(i,j)\in I(z)\times J(z)$ such that (5) holds. Hence, Theorem 10 holds for $x=z$. Thus, Theorem 10 is proven for all $x\in V_n$ with $|I(x)|+|J(x)|=r$. This completes the induction step, and therefore, Theorem 10 is proven.

As an application of Theorem 10, we can now show:

Theorem 11. Let n be a positive integer. Let a_1, a_2, ..., a_n be n nonnegative reals. Let S be a finite set. For every $s\in S$, let r_s be an element of

$(\mathbb{R}^n)^*$ (in other words, a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$), and let b_s be a nonnegative real. Define a function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \sum_{u=1}^{n} a_u |x_u| - \sum_{s \in S} b_s |r_s x|, \qquad \text{where } x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

Then, the following two assertions are equivalent:

Assertion $\mathcal{A}_1$: We have $f(x) \geq 0$ for every $x \in V_n$.

Assertion $\mathcal{A}_2$: We have $f(e_i - e_j) \geq 0$ for any two distinct integers i and j from $\{1, 2, ..., n\}$.

Proof of Theorem 11. We have to prove that the assertions $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent. In other words, we have to prove that $\mathcal{A}_1 \Longrightarrow \mathcal{A}_2$ and $\mathcal{A}_2 \Longrightarrow \mathcal{A}_1$. Actually, $\mathcal{A}_1 \Longrightarrow \mathcal{A}_2$ is trivial (we just have to use that $e_i - e_j \in V_n$ for any two numbers i and j from $\{1, 2, ..., n\}$). It remains to show that $\mathcal{A}_2 \Longrightarrow \mathcal{A}_1$. So assume that Assertion $\mathcal{A}_2$ is valid, i. e. we have $f(e_i - e_j) \geq 0$ for any two distinct integers i and j from $\{1, 2, ..., n\}$. We have to prove that Assertion $\mathcal{A}_1$ holds, i. e. that $f(x) \geq 0$ for every $x \in V_n$.

So let $x \in V_n$ be some vector. According to Theorem 10, there exist nonnegative reals $a_{i,j}$ for all pairs $(i, j) \in I(x) \times J(x)$ such that

$$x = \sum_{(i,j) \in I(x) \times J(x)} a_{i,j} (e_i - e_j).$$

We will now show that

$$|x_u| = \sum_{(i,j) \in I(x) \times J(x)} a_{i,j} \left|(e_i - e_j)_u\right| \qquad \text{for every } u \in \{1, 2, ..., n\}. \tag{6}$$

Here, of course, $(e_i - e_j)_u$ means the u-th coordinate of the vector $e_i - e_j$.

In fact, two cases are possible: the case when $x_u \geq 0$, and the case when $x_u < 0$. We will consider these cases separately.

Case 1: We have $x_u \geq 0$. Then, $|x_u| = x_u$. Hence, in this case, we have $(e_i - e_j)_u \geq 0$ for any two numbers $i \in I(x)$ and $j \in J(x)$ (in fact, $j \in J(x)$ yields $x_j < 0$, so that $u \neq j$ (because $x_j < 0$ and $x_u \geq 0$) and thus $(e_j)_u = 0$, so that $(e_i - e_j)_u = (e_i)_u - (e_j)_u = (e_i)_u - 0 = (e_i)_u = \begin{cases} 1, & \text{if } u = i; \\ 0, & \text{if } u \neq i \end{cases} \geq 0$). Thus, $(e_i - e_j)_u = \left|(e_i - e_j)_u\right|$ for any two numbers $i \in I(x)$ and $j \in J(x)$. Thus,

$$|x_u| = x_u = \sum_{(i,j) \in I(x) \times J(x)} a_{i,j} (e_i - e_j)_u \qquad \left(\text{since } x = \sum_{(i,j) \in I(x) \times J(x)} a_{i,j} (e_i - e_j)\right)$$
$$= \sum_{(i,j) \in I(x) \times J(x)} a_{i,j} \left|(e_i - e_j)_u\right|,$$

and (6) is proven.

Case 2: We have $x_u < 0$. Then, $u \in J(x)$ and $|x_u| = -x_u$. Hence, in this case, we have $(e_i - e_j)_u \leq 0$ for any two numbers $i \in I(x)$ and $j \in J(x)$ (in fact, $i \in I(x)$ yields $x_i > 0$, so that $u \neq i$ (because $x_i > 0$ and $x_u < 0$) and thus $(e_i)_u = 0$, so that $(e_i - e_j)_u = (e_i)_u - (e_j)_u = 0 - (e_j)_u = -(e_j)_u = -\begin{cases} 1, & \text{if } u = j; \\ 0, & \text{if } u \neq j \end{cases} \leq 0$). Thus, $-(e_i - e_j)_u = \left|(e_i - e_j)_u\right|$ for any two numbers $i \in I(x)$ and $j \in J(x)$. Thus,

$$|x_u| = -x_u = -\sum_{(i,j)\in I(x)\times J(x)} a_{i,j}(e_i - e_j)_u \qquad \left(\text{since } x = \sum_{(i,j)\in I(x)\times J(x)} a_{i,j}(e_i - e_j)\right)$$
$$= \sum_{(i,j)\in I(x)\times J(x)} a_{i,j}\left(-(e_i - e_j)_u\right) = \sum_{(i,j)\in I(x)\times J(x)} a_{i,j}\left|(e_i - e_j)_u\right|,$$

and (6) is proven.

Hence, in both cases, (6) is proven. Thus, (6) always holds. Now let us continue our proof of $\mathcal{A}_2 \Longrightarrow \mathcal{A}_1$:

We have

$$\sum_{s\in S} b_s |r_s x| = \sum_{s\in S} b_s \left| r_s \sum_{(i,j)\in I(x)\times J(x)} a_{i,j}(e_i - e_j)\right| \qquad \left(\text{since } x = \sum_{(i,j)\in I(x)\times J(x)} a_{i,j}(e_i - e_j)\right)$$
$$= \sum_{s\in S} b_s \left|\sum_{(i,j)\in I(x)\times J(x)} a_{i,j} r_s (e_i - e_j)\right|$$
$$\leq \sum_{s\in S} b_s \sum_{(i,j)\in I(x)\times J(x)} a_{i,j} |r_s (e_i - e_j)|$$
$$\text{(by the triangle inequality, since all } a_{i,j} \text{ and all } b_s \text{ are nonnegative)}.$$

Thus,

$$f(x) = \sum_{u=1}^{n} a_u |x_u| - \sum_{s\in S} b_s |r_s x| \geq \sum_{u=1}^{n} a_u |x_u| - \sum_{s\in S} b_s \sum_{(i,j)\in I(x)\times J(x)} a_{i,j} |r_s (e_i - e_j)|$$
$$= \sum_{u=1}^{n} a_u \cdot \sum_{(i,j)\in I(x)\times J(x)} a_{i,j}\left|(e_i - e_j)_u\right| - \sum_{s\in S} b_s \sum_{(i,j)\in I(x)\times J(x)} a_{i,j} |r_s (e_i - e_j)| \qquad \text{(by (6))}$$
$$= \sum_{(i,j)\in I(x)\times J(x)} a_{i,j} \sum_{u=1}^{n} a_u \left|(e_i - e_j)_u\right| - \sum_{(i,j)\in I(x)\times J(x)} a_{i,j} \sum_{s\in S} b_s |r_s (e_i - e_j)|$$
$$= \sum_{(i,j)\in I(x)\times J(x)} a_{i,j} \cdot \left(\sum_{u=1}^{n} a_u \left|(e_i - e_j)_u\right| - \sum_{s\in S} b_s |r_s (e_i - e_j)|\right)$$
$$= \sum_{(i,j)\in I(x)\times J(x)} \underbrace{a_{i,j}}_{\geq 0} \cdot \underbrace{f(e_i - e_j)}_{\geq 0} \geq 0.$$

(Here, $f(e_i - e_j) \geq 0$ because i and j are two distinct integers from $\{1, 2, ..., n\}$; in fact, i and j are distinct because $i \in I(x)$ and $j \in J(x)$, and the sets $I(x)$ and $J(x)$ are disjoint.)

Hence, we have obtained $f(x) \geq 0$. This proves the assertion $\mathcal{A}_1$. Therefore, the implication $\mathcal{A}_2 \Longrightarrow \mathcal{A}_1$ is proven, and the proof of Theorem 11 is complete.

5. Restating Theorem 11

Now we consider a result which follows from Theorem 11 pretty obviously (but again, formalizing the proof is going to be gruelling):

Theorem 12. Let n be a nonnegative integer. Let a_1, a_2, ..., a_n and a be $n+1$ nonnegative reals. Let S be a finite set. For every $s \in S$, let r_s be an element of $(\mathbb{R}^n)^*$ (in other words, a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$), and let b_s be a nonnegative real. Define a function $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) = \sum_{u=1}^{n} a_u |x_u| + a |x_1 + x_2 + ... + x_n| - \sum_{s \in S} b_s |r_s x|, \qquad \text{where } x = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

Then, the following two assertions are equivalent:

Assertion $\mathcal{B}_1$: We have $g(x) \geq 0$ for every $x \in \mathbb{R}^n$.

Assertion $\mathcal{B}_2$: We have $g(e_i) \geq 0$ for every integer $i \in \{1, 2, ..., n\}$, and $g(e_i - e_j) \geq 0$ for any two distinct integers i and j from $\{1, 2, ..., n\}$.

Proof of Theorem 12. We are going to restate Theorem 12 before we actually prove it. But first, we introduce a notation:

Let $(\widetilde{e_1}, \widetilde{e_2}, ..., \widetilde{e_{n-1}})$ be the standard basis of the vector space $\mathbb{R}^{n-1}$; in other words, for every $i \in \{1, 2, ..., n-1\}$, let $\widetilde{e_i}$ be the vector from $\mathbb{R}^{n-1}$ such that $(\widetilde{e_i})_i = 1$ and $(\widetilde{e_i})_j = 0$ for every $j \in \{1, 2, ..., n-1\} \setminus \{i\}$.

Now we will restate Theorem 12 by renaming n into $n-1$ (thus replacing e_i by $\widetilde{e_i}$ as well) and a into a_n:

Theorem 12b. Let n be a positive integer. Let a_1, a_2, ..., a_{n-1}, a_n be n nonnegative reals. Let S be a finite set. For every $s \in S$, let r_s be an element of $(\mathbb{R}^{n-1})^*$ (in other words, a linear transformation from $\mathbb{R}^{n-1}$ to $\mathbb{R}$), and let b_s be a nonnegative real. Define a function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$g(x) = \sum_{u=1}^{n-1} a_u |x_u| + a_n |x_1 + x_2 + ... + x_{n-1}| - \sum_{s \in S} b_s |r_s x|, \qquad \text{where } x = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_{n-1} \end{pmatrix} \in \mathbb{R}^{n-1}.$$

Then, the following two assertions are equivalent:

Assertion $\mathcal{C}_1$: We have $g(x) \geq 0$ for every $x \in \mathbb{R}^{n-1}$.

Assertion $\mathcal{C}_2$: We have $g(\widetilde{e_i}) \geq 0$ for every integer $i \in \{1, 2, ..., n-1\}$, and $g(\widetilde{e_i} - \widetilde{e_j}) \geq 0$ for any two distinct integers i and j from $\{1, 2, ..., n-1\}$.

Theorem 12b is equivalent to Theorem 12 (because Theorem 12b is just Theorem 12, applied to $n-1$ instead of n). Thus, proving Theorem 12b will be enough to verify Theorem 12.

Proof of Theorem 12b. In order to establish Theorem 12b, we have to prove that the assertions $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent. In other words, we have to verify the two implications $\mathcal{C}_1 \Longrightarrow \mathcal{C}_2$ and $\mathcal{C}_2 \Longrightarrow \mathcal{C}_1$.

The implication $\mathcal{C}_1 \Longrightarrow \mathcal{C}_2$ is absolutely trivial. Hence, it only remains to prove the implication $\mathcal{C}_2 \Longrightarrow \mathcal{C}_1$.

So assume that the assertion $\mathcal{C}_2$ holds, i. e. that we have $g(\widetilde{e_i}) \geq 0$ for every integer $i \in \{1, 2, ..., n-1\}$, and $g(\widetilde{e_i} - \widetilde{e_j}) \geq 0$ for any two distinct integers i and j from $\{1, 2, ..., n-1\}$. We want to show that Assertion $\mathcal{C}_1$ holds, i. e. that $g(x) \geq 0$ is satisfied for every $x \in \mathbb{R}^{n-1}$.

Since $(\widetilde{e_1}, \widetilde{e_2}, ..., \widetilde{e_{n-1}})$ is the standard basis of the vector space $\mathbb{R}^{n-1}$, every vector $x \in \mathbb{R}^{n-1}$ satisfies $x = \sum\limits_{i=1}^{n-1} x_i \widetilde{e_i}$.

Since $(e_1, e_2, ..., e_n)$ is the standard basis of the vector space $\mathbb{R}^n$, every vector $x \in \mathbb{R}^n$ satisfies $x = \sum\limits_{i=1}^{n} x_i e_i$.

Let $\phi_n : \mathbb{R}^{n-1} \to \mathbb{R}^n$ be the linear transformation defined by $\phi_n \widetilde{e_i} = e_i - e_n$ for every $i \in \{1, 2, ..., n-1\}$. (This linear transformation is uniquely defined this way because $(\widetilde{e_1}, \widetilde{e_2}, ..., \widetilde{e_{n-1}})$ is a basis of $\mathbb{R}^{n-1}$.) For every $x \in \mathbb{R}^{n-1}$, we then have

$$\begin{aligned}
\phi_n x &= \phi_n \left(\sum_{i=1}^{n-1} x_i \widetilde{e_i} \right) = \sum_{i=1}^{n-1} x_i \phi_n \widetilde{e_i} \qquad \text{(since } \phi_n \text{ is linear)} \\
&= \sum_{i=1}^{n-1} x_i (e_i - e_n) = \sum_{i=1}^{n-1} x_i e_i - \sum_{i=1}^{n-1} x_i e_n = \sum_{i=1}^{n-1} x_i e_i - (x_1 + x_2 + ... + x_{n-1}) e_n \\
&= \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_{n-1} \\ -(x_1 + x_2 + ... + x_{n-1}) \end{pmatrix}, \qquad (7)
\end{aligned}$$

As a consequence of this computation, we get $\phi_n x \in V_n$ for every $x \in \mathbb{R}^{n-1}$ (in fact, above we have shown that $\phi_n x = \sum\limits_{i=1}^{n-1} x_i (e_i - e_n)$; but since $e_i - e_n \in V_n$ for every $i \in \{1, 2, ..., n-1\}$, we must have $\sum\limits_{i=1}^{n-1} x_i (e_i - e_n) \in V_n$, so that $\phi_n x \in V_n$). Hence, $\operatorname{Im} \phi_n \subseteq V_n$.

Let $\psi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the linear transformation defined by $\psi_n e_i = \begin{cases} \widetilde{e_i}, & \text{if } i \in \{1, 2, ..., n-1\}; \\ 0, & \text{if } i = n \end{cases}$ for every $i \in \{1, 2, ..., n\}$. (This linear transformation is uniquely defined this way because $(e_1, e_2, ..., e_n)$ is a basis of $\mathbb{R}^n$.) For every $x \in \mathbb{R}^n$, we then have

$$\psi_n x = \psi_n \left(\sum_{i=1}^{n} x_i e_i \right) = \sum_{i=1}^{n} x_i \psi_n e_i \qquad \text{(since } \psi_n \text{ is linear)}$$
$$= \sum_{i=1}^{n} x_i \begin{cases} \widetilde{e_i}, & \text{if } i \in \{1, 2, ..., n-1\}; \\ 0, & \text{if } i = n \end{cases}$$
$$= \sum_{i=1}^{n-1} x_i \widetilde{e_i} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_{n-1} \end{pmatrix}.$$

Then, $\psi_n \phi_n = \text{id}$ (in fact, for every $i \in \{1, 2, ..., n-1\}$, we have

$$\psi_n \phi_n \widetilde{e_i} = \psi_n (e_i - e_n) = \psi_n e_i - \psi_n e_n \qquad \text{(since } \psi_n \text{ is linear)}$$
$$= \widetilde{e_i} - 0 = \widetilde{e_i};$$

thus, for every $x \in \mathbb{R}^{n-1}$, we have

$$\psi_n \phi_n x = \psi_n \phi_n \left(\sum_{i=1}^{n-1} x_i \widetilde{e_i} \right) = \sum_{i=1}^{n-1} x_i \psi_n \phi_n \widetilde{e_i}$$
(since the function $\psi_n \phi_n$ is linear, because ψ_n and ϕ_n are linear)
$$= \sum_{i=1}^{n-1} x_i \widetilde{e_i} = x,$$

and therefore $\psi_n \phi_n = \text{id}$).

We define a function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \sum_{u=1}^{n} a_u |x_u| - \sum_{s \in S} b_s |r_s \psi_n x|, \qquad \text{where } x = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

Note that

$$f(-x) = f(x) \qquad \text{for every } x \in \mathbb{R}^n, \tag{8}$$

since

$$f(-x) = \sum_{u=1}^{n} a_u |(-x)_u| - \sum_{s \in S} b_s |r_s \psi_n (-x)| = \sum_{u=1}^{n} a_u |-x_u| - \sum_{s \in S} b_s |-r_s \psi_n x|$$
(here, we have $r_s \psi_n (-x) = -r_s \psi_n x$ since r_s and ψ_n are linear functions)
$$= \sum_{u=1}^{n} a_u |x_u| - \sum_{s \in S} b_s |r_s \psi_n x| = f(x).$$

Furthermore, I claim that

$$f(\phi_n x) = g(x) \qquad \text{for every } x \in \mathbb{R}^{n-1}. \tag{9}$$

In order to prove this, we note that (7) yields $(\phi_n x)_u = x_u$ for all $u \in \{1, 2, ..., n-1\}$ and $(\phi_n x)_n = -(x_1 + x_2 + ... + x_{n-1})$, while $\psi_n \phi_n = \text{id}$ yields $\psi_n \phi_n x = x$, so that

$$\begin{aligned} f(\phi_n x) &= \sum_{u=1}^{n} a_u \left|(\phi_n x)_u\right| - \sum_{s\in S} b_s \left|r_s \psi_n \phi_n x\right| \\ &= \sum_{u=1}^{n-1} a_u \left|(\phi_n x)_u\right| + a_n \left|(\phi_n x)_n\right| - \sum_{s\in S} b_s \left|r_s \psi_n \phi_n x\right| \\ &= \sum_{u=1}^{n-1} a_u \left|x_u\right| + a_n \left|-(x_1 + x_2 + ... + x_{n-1})\right| - \sum_{s\in S} b_s \left|r_s x\right| \\ &\qquad \left(\text{since } (\phi_n x)_u = x_u \text{ for all } u \in \{1, 2, ..., n-1\} \text{ and } \right. \\ &\qquad \left. (\phi_n x)_n = -(x_1 + x_2 + ... + x_{n-1}), \text{ and } \psi_n \phi_n x = x\right) \\ &= \sum_{u=1}^{n-1} a_u \left|x_u\right| + a_n \left|x_1 + x_2 + ... + x_{n-1}\right| - \sum_{s\in S} b_s \left|r_s x\right| = g(x), \end{aligned}$$

and thus (9) is proven.

Now, we are going to show that

$$f(e_i - e_j) \geq 0 \text{ for any two distinct integers } i \text{ and } j \text{ from } \{1, 2, ..., n\}. \qquad (10)$$

In order to prove (10), we distinguish between three different cases:

Case 1: We have $i \in \{1, 2, ..., n-1\}$ and $j \in \{1, 2, ..., n-1\}$.

Case 2: We have $i \in \{1, 2, ..., n-1\}$ and $j = n$.

Case 3: We have $i = n$ and $j \in \{1, 2, ..., n-1\}$.

(In fact, the case when both $i = n$ and $j = n$ cannot occur, since i and j must be distinct).

In Case 1, we have

$$\begin{aligned} f(e_i - e_j) &= f((e_i - e_n) - (e_j - e_n)) = f\left(\phi_n \widetilde{e_i} - \phi_n \widetilde{e_j}\right) \\ &= f\left(\phi_n \left(\widetilde{e_i} - \widetilde{e_j}\right)\right) \qquad \left(\text{since } \phi_n \widetilde{e_i} - \phi_n \widetilde{e_j} = \phi_n \left(\widetilde{e_i} - \widetilde{e_j}\right), \text{ because } \phi_n \text{ is linear}\right) \\ &= g\left(\widetilde{e_i} - \widetilde{e_j}\right) \qquad (\text{after } (9)) \\ &\geq 0 \qquad (\text{by assumption}). \end{aligned}$$

In Case 2, we have

$$\begin{aligned} f(e_i - e_j) &= f(e_i - e_n) = f\left(\phi_n \widetilde{e_i}\right) = g\left(\widetilde{e_i}\right) \qquad (\text{after } (9)) \\ &\geq 0 \qquad (\text{by assumption}). \end{aligned}$$

In Case 3, we have

$$\begin{aligned} f(e_i - e_j) &= f(e_n - e_j) = f(-(e_j - e_n)) = f(e_j - e_n) \qquad (\text{after } (8)) \\ &= f\left(\phi_n \widetilde{e_j}\right) = g\left(\widetilde{e_j}\right) \qquad (\text{after } (9)) \\ &\geq 0 \qquad (\text{by assumption}). \end{aligned}$$

Thus, $f(e_i - e_j) \geq 0$ holds in all three possible cases. Hence, (10) is proven.

Now, our function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$f(x) = \sum_{u=1}^{n} a_u |x_u| - \sum_{s \in S} b_s |r_s \psi_n x|, \qquad \text{where } x = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

Here, n is a positive integer; the numbers a_1, a_2, ..., a_n are n nonnegative reals; the set S is a finite set; for every $s \in S$, the function $r_s \psi_n$ is an element of $(\mathbb{R}^n)^*$ (in other words, a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$), and b_s is a nonnegative real.

Hence, we can apply Theorem 11 to our function f, and we obtain that for our function f, the Assertions $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent. In other words, our function f satisfies Assertion $\mathcal{A}_1$ if and only if it satisfies Assertion $\mathcal{A}_2$.

Now, according to (10), our function f satisfies Assertion $\mathcal{A}_2$. Thus, this function f must also satisfy Assertion $\mathcal{A}_1$. In other words, $f(x) \geq 0$ holds for every $x \in V_n$. Hence, $f(\phi_n x) \geq 0$ holds for every $x \in \mathbb{R}^{n-1}$ (because $\phi_n x \in V_n$, since $\operatorname{Im} \phi_n \subseteq V_n$). Since $f(\phi_n x) = g(x)$ according to (9), we have therefore proven that $g(x) \geq 0$ holds for every $x \in \mathbb{R}^{n-1}$. Hence, Assertion $\mathcal{C}_1$ is proven. Thus, we have showed that $\mathcal{C}_2 \Longrightarrow \mathcal{C}_1$, and thus the proof of Theorem 12b is complete.

Since Theorem 12b is equivalent to Theorem 12, this also proves Theorem 12.

As if this wasn't enough, here comes a further restatement of Theorem 12:

Theorem 13. Let n be a nonnegative integer. Let a_1, a_2, ..., a_n and a be $n+1$ nonnegative reals. Let S be a finite set. For every $s \in S$, let $r_{s,1}$, $r_{s,2}$, ..., $r_{s,n}$ be n nonnegative reals, and let b_s be a nonnegative real. Assume that the following two conditions hold:

$$a_i + a \geq \sum_{s \in S} b_s r_{s,i} \qquad \text{for every } i \in \{1, 2, ..., n\};$$

$$a_i + a_j \geq \sum_{s \in S} b_s |r_{s,i} - r_{s,j}| \qquad \text{for any two distinct integers } i \text{ and } j \text{ from } \{1, 2, ..., n\}.$$

Let y_1, y_2, ..., y_n be n reals. Then,

$$\sum_{i=1}^{n} a_i |y_i| + a \left| \sum_{v=1}^{n} y_v \right| - \sum_{s \in S} b_s \left| \sum_{v=1}^{n} r_{s,v} y_v \right| \geq 0.$$

Proof of Theorem 13. For every $s \in S$, let $r_s = (r_{s,1}, r_{s,2}, ..., r_{s,n}) \in (\mathbb{R}^n)^*$ be the n-dimensional covector whose i-th coordinate is $r_{s,i}$ for every $i \in \{1, 2, ..., n\}$. Define a function $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) = \sum_{u=1}^{n} a_u |x_u| + a |x_1 + x_2 + ... + x_n| - \sum_{s \in S} b_s |r_s x|, \qquad \text{where } x = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

For every $i \in \{1, 2, ..., n\}$, we have $(e_i)_u = \begin{cases} 1, & \text{if } u = i; \\ 0, & \text{if } u \neq i \end{cases}$ for all $u \in \{1, 2, ..., n\}$, so that $(e_i)_1 + (e_i)_2 + ... + (e_i)_n = 1$, and for every $s \in S$, we have

$$r_s e_i = \sum_{u=1}^{n} r_{s,u} (e_i)_u \qquad \left(\text{since } r_s = (r_{s,1}, r_{s,2}, ..., r_{s,n})\right)$$
$$= \sum_{u=1}^{n} r_{s,u} \begin{cases} 1, & \text{if } u = i; \\ 0, & \text{if } u \neq i \end{cases} = r_{s,i},$$

so that

$$g(e_i) = \sum_{u=1}^{n} a_u \left|(e_i)_u\right| + a \left|(e_i)_1 + (e_i)_2 + ... + (e_i)_n\right| - \sum_{s \in S} b_s \left|r_s e_i\right|$$
$$= \sum_{u=1}^{n} a_u \left| \begin{cases} 1, & \text{if } u = i; \\ 0, & \text{if } u \neq i \end{cases} \right| + a\left|1\right| - \sum_{s \in S} b_s \left|r_{s,i}\right|$$
$$= a_i \left|1\right| + a\left|1\right| - \sum_{s \in S} b_s \underbrace{\left|r_{s,i}\right|}_{\substack{=r_{s,i}, \\ \text{since} \\ r_{s,i} \geq 0}} = a_i + a - \sum_{s \in S} b_s r_{s,i} \geq 0$$

(since $a_i + a \geq \sum\limits_{s \in S} b_s r_{s,i}$ by the conditions of Theorem 13).

For any two distinct integers i and j from $\{1, 2, ..., n\}$, we have $(e_i - e_j)_u = \begin{cases} 1, & \text{if } u = i; \\ -1, & \text{if } u = j; \\ 0, & \text{if } u \neq i \text{ and } u \neq j \end{cases}$ for all $u \in \{1, 2, ..., n\}$, so that $(e_i - e_j)_1 + (e_i - e_j)_2 + ... + (e_i - e_j)_n = 0$, and for every $s \in S$, we have

$$r_s (e_i - e_j) = \sum_{u=1}^{n} r_{s,u} (e_i - e_j)_u \qquad \left(\text{since } r_s = (r_{s,1}, r_{s,2}, ..., r_{s,n})\right)$$
$$= \sum_{u=1}^{n} r_{s,u} \begin{cases} 1, & \text{if } u = i; \\ -1, & \text{if } u = j; \\ 0, & \text{if } u \neq i \text{ and } u \neq j \end{cases} = r_{s,i} - r_{s,j},$$

and thus

$$g(e_i - e_j) = \sum_{u=1}^{n} a_u \left|(e_i - e_j)_u\right| + a \left|(e_i - e_j)_1 + (e_i - e_j)_2 + ... + (e_i - e_j)_n\right| - \sum_{s \in S} b_s \left|r_s (e_i - e_j)\right|$$
$$= \sum_{u=1}^{n} a_u \left| \begin{cases} 1, & \text{if } u = i; \\ -1, & \text{if } u = j; \\ 0, & \text{if } u \neq i \text{ and } u \neq j \end{cases} \right| + a\left|0\right| - \sum_{s \in S} b_s \left|r_{s,i} - r_{s,j}\right|$$
$$= (a_i \left|1\right| + a_j \left|-1\right|) + a\left|0\right| - \sum_{s \in S} b_s \left|r_{s,i} - r_{s,j}\right| = (a_i + a_j) + 0 - \sum_{s \in S} b_s \left|r_{s,i} - r_{s,j}\right|$$
$$= a_i + a_j - \sum_{s \in S} b_s \left|r_{s,i} - r_{s,j}\right| \geq 0$$

(since $a_i + a_j \geq \sum\limits_{s \in S} b_s \left|r_{s,i} - r_{s,j}\right|$ by the condition of Theorem 13).

So we have shown that $g(e_i) \geq 0$ for every integer $i \in \{1, 2, ..., n\}$, and $g(e_i - e_j) \geq 0$ for any two distinct integers i and j from $\{1, 2, ..., n\}$. Thus, Assertion $\mathcal{B}_2$ of Theorem 12 is fulfilled. According to Theorem 12, the assertions $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent, so that Assertion $\mathcal{B}_1$ must be fulfilled as well. Hence, $g(x) \geq 0$ for every $x \in \mathbb{R}^n$. In particular, if we set $x = \begin{pmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{pmatrix}$, then $r_s x = \sum_{v=1}^{n} r_{s,v} y_v$ (since $r_s = (r_{s,1}, r_{s,2}, ..., r_{s,n})$), so that

$$\begin{aligned} g(x) &= \sum_{u=1}^{n} a_u |y_u| + a |y_1 + y_2 + ... + y_n| - \sum_{s \in S} b_s |r_s x| \\ &= \sum_{u=1}^{n} a_u |y_u| + a |y_1 + y_2 + ... + y_n| - \sum_{s \in S} b_s \left| \sum_{v=1}^{n} r_{s,v} y_v \right| \\ &= \sum_{i=1}^{n} a_i |y_i| + a \left| \sum_{v=1}^{n} y_v \right| - \sum_{s \in S} b_s \left| \sum_{v=1}^{n} r_{s,v} y_v \right|, \end{aligned}$$

and thus $g(x) \geq 0$ yields

$$\sum_{i=1}^{n} a_i |y_i| + a \left| \sum_{v=1}^{n} y_v \right| - \sum_{s \in S} b_s \left| \sum_{v=1}^{n} r_{s,v} y_v \right| \geq 0.$$

Theorem 13 is thus proven.

6. A general condition for Popoviciu-like inequalities

Now, we state a result more general than Theorem 5b:

Theorem 14. Let n be a nonnegative integer. Let $a_1, a_2, ..., a_n$ and a be $n+1$ nonnegative reals. Let S be a finite set. For every $s \in S$, let $r_{s,1}, r_{s,2}, ..., r_{s,n}$ be n nonnegative reals, and let b_s be a nonnegative real. Assume that the following two conditions hold[2]:

$$a_i + a = \sum_{s \in S} b_s r_{s,i} \qquad \text{for every } i \in \{1, 2, ..., n\};$$

$$a_i + a_j \geq \sum_{s \in S} b_s |r_{s,i} - r_{s,j}| \qquad \text{for any two distinct integers } i \text{ and } j \text{ from } \{1, 2, ..., n\}.$$

Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Let $w_1, w_2, ..., w_n$ be nonnegative reals. Assume that $\sum_{v=1}^{n} w_v \neq 0$ and $\sum_{v=1}^{n} r_{s,v} w_v \neq 0$ for all $s \in S$.

[2]The second of these two conditions ($a_i + a_j \geq \sum_{s \in S} b_s |r_{s,i} - r_{s,j}|$ for any two distinct integers i and j from $\{1, 2, ..., n\}$) is identic with the second assumed condition in Theorem 13, but the first one ($a_i + a = \sum_{s \in S} b_s r_{s,i}$ for every $i \in \{1, 2, ..., n\}$) is *stronger* than the first required condition in Theorem 13 (which only said that $a_i + a \geq \sum_{s \in S} b_s r_{s,i}$ for every $i \in \{1, 2, ..., n\}$).

Let x_1, x_2, ..., x_n be n points from the interval I. Then, the inequality

$$\sum_{i=1}^{n} a_i w_i f(x_i) + a\left(\sum_{v=1}^{n} w_v\right) f\left(\frac{\sum\limits_{v=1}^{n} w_v x_v}{\sum\limits_{v=1}^{n} w_v}\right) \geq \sum_{s\in S} b_s \left(\sum_{v=1}^{n} r_{s,v} w_v\right) f\left(\frac{\sum\limits_{v=1}^{n} r_{s,v} w_v x_v}{\sum\limits_{v=1}^{n} r_{s,v} w_v}\right)$$

holds.

Remark. Written in a less formal way, this inequality states that

$$\sum_{i=1}^{n} a_i w_i f(x_i) + a(w_1 + w_2 + ... + w_n) f\left(\frac{w_1 x_1 + w_2 x_2 + ... + w_n x_n}{w_1 + w_2 + ... + w_n}\right)$$
$$\geq \sum_{s\in S} b_s (r_{s,1} w_1 + r_{s,2} w_2 + ... + r_{s,n} w_n) f\left(\frac{r_{s,1} w_1 x_1 + r_{s,2} w_2 x_2 + ... + r_{s,n} w_n x_n}{r_{s,1} w_1 + r_{s,2} w_2 + ... + r_{s,n} w_n}\right).$$

Proof of Theorem 14. Since the elements of the finite set S are used as labels only, we can assume without loss of generality that $S = \{n+2, n+3, ..., N\}$ for some integer $N \geq n+1$ (we just rename the elements of S into $n+2$, $n+3$, ..., N, where $N = n + 1 + |S|$; this is possible because the set S is finite[3]). Define

$$u_i = a_i w_i \qquad \text{for all } i \in \{1, 2, ..., n\};$$
$$u_{n+1} = a\left(\sum_{v=1}^{n} w_v\right);$$
$$u_s = -b_s \left(\sum_{v=1}^{n} r_{s,v} w_v\right) \qquad \text{for all } s \in \{n+2, n+3, ..., N\} \text{ (that is, for all } s \in S).$$

Also define

$$z_i = x_i \qquad \text{for all } i \in \{1, 2, ..., n\};$$
$$z_{n+1} = \frac{\sum\limits_{v=1}^{n} w_v x_v}{\sum\limits_{v=1}^{n} w_v};$$
$$z_s = \frac{\sum\limits_{v=1}^{n} r_{s,v} w_v x_v}{\sum\limits_{v=1}^{n} r_{s,v} w_v} \qquad \text{for all } s \in \{n+2, n+3, ..., N\} \text{ (that is, for all } s \in S).$$

Each of these N reals z_1, z_2, ..., z_N is a weighted mean of the reals x_1, x_2, ..., x_n with nonnegative weights[4]. Since the reals x_1, x_2, ..., x_n lie in the interval I, we can

[3] In particular, $N = n + 1$ if $S = \varnothing$.

[4] In fact,

- for z_1, z_2, ..., z_n, this is clear because $z_i = x_i = \frac{0x_1 + 0x_2 + ... + 0x_{i-1} + 1x_i + 0x_{i+1} + 0x_{i+2} + ... + 0x_n}{0 + 0 + ... + 0 + 1 + 0 + 0 + ... + 0}$ for all $i \in \{1, 2, ..., n\}$;

thus conclude that each of the N reals z_1, z_2, ..., z_N lies in the interval I as well (because if some reals lie in some interval I, then any weighted mean of these reals with nonnegative weights must also lie in I). In other words, the points z_1, z_2, ..., z_N are N points from I.

Now,

$$\sum_{i=1}^{n} a_i w_i f\left(x_i\right) + a\left(\sum_{v=1}^{n} w_v\right) f\left(\frac{\sum\limits_{v=1}^{n} w_v x_v}{\sum\limits_{v=1}^{n} w_v}\right) - \sum_{s\in S} b_s \left(\sum_{v=1}^{n} r_{s,v} w_v\right) f\left(\frac{\sum\limits_{v=1}^{n} r_{s,v} w_v x_v}{\sum\limits_{v=1}^{n} r_{s,v} w_v}\right)$$

$$= \sum_{i=1}^{n} \underbrace{a_i w_i}_{=u_i} f\left(\underbrace{x_i}_{=z_i}\right) + \underbrace{a\left(\sum_{v=1}^{n} w_v\right)}_{=u_{n+1}} f\left(\underbrace{\frac{\sum\limits_{v=1}^{n} w_v x_v}{\sum\limits_{v=1}^{n} w_v}}_{=z_{n+1}}\right) + \sum_{s\in S} \left(\underbrace{-b_s \left(\sum_{v=1}^{n} r_{s,v} w_v\right)}_{=u_s}\right) f\left(\underbrace{\frac{\sum\limits_{v=1}^{n} r_{s,v} w_v x_v}{\sum\limits_{v=1}^{n} r_{s,v} w_v}}_{=z_s}\right)$$

$$= \sum_{i=1}^{n} u_i f\left(z_i\right) + u_{n+1} f\left(z_{n+1}\right) + \sum_{s\in S} u_s f\left(z_s\right) = \sum_{i=1}^{n} u_i f\left(z_i\right) + u_{n+1} f\left(z_{n+1}\right) + \sum_{s=n+2}^{N} u_s f\left(z_s\right)$$

$$= \sum_{k=1}^{N} u_k f\left(z_k\right).$$

Hence, once we are able to show that $\sum\limits_{k=1}^{N} u_k f\left(z_k\right) \geq 0$, we will obtain

$$\sum_{i=1}^{n} a_i w_i f\left(x_i\right) + a\left(\sum_{v=1}^{n} w_v\right) f\left(\frac{\sum\limits_{v=1}^{n} w_v x_v}{\sum\limits_{v=1}^{n} w_v}\right) \geq \sum_{s\in S} b_s \left(\sum_{v=1}^{n} r_{s,v} w_v\right) f\left(\frac{\sum\limits_{v=1}^{n} r_{s,v} w_v x_v}{\sum\limits_{v=1}^{n} r_{s,v} w_v}\right),$$

and thus Theorem 14 will be established.

Therefore, in order to prove Theorem 14, it remains to prove the inequality $\sum\limits_{k=1}^{N} u_k f\left(z_k\right) \geq 0$.

- for z_{n+1}, this is clear from $z_{n+1} = \dfrac{\sum\limits_{v=1}^{n} w_v x_v}{\sum\limits_{v=1}^{n} w_v}$;
- for z_{n+2}, z_{n+3}, ..., z_N, this is clear because $z_s = \dfrac{\sum\limits_{v=1}^{n} r_{s,v} w_v x_v}{\sum\limits_{v=1}^{n} r_{s,v} w_v}$ for all $s \in \{n+2, n+3, ..., N\}$.

We have

$$\begin{aligned}
\sum_{k=1}^{N} u_k &= \sum_{i=1}^{n} u_i + u_{n+1} + \sum_{s=n+2}^{N} u_s = \sum_{i=1}^{n} u_i + u_{n+1} + \sum_{s\in S} u_s \\
&= \sum_{i=1}^{n} a_i w_i + a\left(\sum_{v=1}^{n} w_v\right) + \sum_{s\in S}\left(-b_s\left(\sum_{v=1}^{n} r_{s,v} w_v\right)\right) \\
&= \sum_{i=1}^{n} a_i w_i + a\left(\sum_{i=1}^{n} w_i\right) + \sum_{s\in S}\left(-b_s\left(\sum_{i=1}^{n} r_{s,i} w_i\right)\right) \\
&= \sum_{i=1}^{n} a_i w_i + \sum_{i=1}^{n} a w_i - \sum_{i=1}^{n}\sum_{s\in S} b_s r_{s,i} w_i = \sum_{i=1}^{n}\left(a_i w_i + a w_i - \sum_{s\in S} b_s r_{s,i} w_i\right) \\
&= \sum_{i=1}^{n}\left(a_i + a - \sum_{s\in S} b_s r_{s,i}\right) w_i \\
&= \sum_{i=1}^{n} 0 w_i \qquad \left(\begin{array}{c}\text{since } a_i + a = \sum_{s\in S} b_s r_{s,i} \text{ by an assumption of Theorem 14,} \\ \text{and thus } a_i + a - \sum_{s\in S} b_s r_{s,i} = 0\end{array}\right) \\
&= 0.
\end{aligned}$$

Next, we are going to prove that $\sum_{k=1}^{N} u_k \left|z_k - t\right| \geq 0$ holds for every $t \in \{z_1, z_2, ..., z_N\}$. In fact, let $t \in \{z_1, z_2, ..., z_N\}$ be arbitrary. Set $y_i = w_i (x_i - t)$ for every $i \in \{1, 2, ..., n\}$. Then, for all $i \in \{1, 2, ..., n\}$, we have $w_i (z_i - t) = w_i (x_i - t) = y_i$. Furthermore,

$$z_{n+1} - t = \frac{\sum_{v=1}^{n} w_v x_v}{\sum_{v=1}^{n} w_v} - t = \frac{\sum_{v=1}^{n} w_v x_v - \sum_{v=1}^{n} w_v \cdot t}{\sum_{v=1}^{n} w_v} = \frac{\sum_{v=1}^{n} w_v (x_v - t)}{\sum_{v=1}^{n} w_v} = \frac{\sum_{v=1}^{n} y_v}{\sum_{v=1}^{n} w_v}.$$

Finally, for all $s \in \{n+2, n+3, ..., N\}$ (that is, for all $s \in S$), we have

$$z_s - t = \frac{\sum_{v=1}^{n} r_{s,v} w_v x_v}{\sum_{v=1}^{n} r_{s,v} w_v} - t = \frac{\sum_{v=1}^{n} r_{s,v} w_v x_v - \sum_{v=1}^{n} r_{s,v} w_v \cdot t}{\sum_{v=1}^{n} r_{s,v} w_v} = \frac{\sum_{v=1}^{n} r_{s,v} w_v (x_v - t)}{\sum_{v=1}^{n} r_{s,v} w_v} = \frac{\sum_{v=1}^{n} r_{s,v} y_v}{\sum_{v=1}^{n} r_{s,v} w_v}.$$

Hence,

$$\sum_{k=1}^{N} u_k |z_k - t| = \sum_{i=1}^{n} u_i |z_i - t| + u_{n+1} |z_{n+1} - t| + \sum_{s=n+2}^{N} u_s |z_s - t|$$

$$= \sum_{i=1}^{n} a_i \underbrace{w_i |z_i - t|}_{\substack{=|w_i(z_i-t)|,\\ \text{since } w_i \geq 0}} + a \left(\sum_{v=1}^{n} w_v \right) \left| \frac{\sum\limits_{v=1}^{n} y_v}{\sum\limits_{v=1}^{n} w_v} \right| + \sum_{s=n+2}^{N} \left(-b_s \left(\sum_{v=1}^{n} r_{s,v} w_v \right) \right) \left| \frac{\sum\limits_{v=1}^{n} r_{s,v} y_v}{\sum\limits_{v=1}^{n} r_{s,v} w_v} \right|$$

$$= \sum_{i=1}^{n} a_i |w_i (z_i - t)| + a \left(\sum_{v=1}^{n} w_v \right) \frac{\left| \sum\limits_{v=1}^{n} y_v \right|}{\sum\limits_{v=1}^{n} w_v} + \sum_{s=n+2}^{N} \left(-b_s \left(\sum_{v=1}^{n} r_{s,v} w_v \right) \right) \frac{\left| \sum\limits_{v=1}^{n} r_{s,v} y_v \right|}{\sum\limits_{v=1}^{n} r_{s,v} w_v}$$

$$\left(\begin{array}{c} \text{here we have pulled the } \sum\limits_{v=1}^{n} w_v \text{ and } \sum\limits_{v=1}^{n} r_{s,v} w_v \text{ terms out of the modulus} \\ \text{signs, since they are positive (in fact, they are } \neq 0 \text{ by an assumption} \\ \text{of Theorem 14, and nonnegative because } w_i \text{ and } r_{s,i} \text{ are all nonnegative)} \end{array} \right)$$

$$= \sum_{i=1}^{n} a_i |y_i| + a \left| \sum_{v=1}^{n} y_v \right| + \sum_{s=n+2}^{N} (-b_s) \left| \sum_{v=1}^{n} r_{s,v} y_v \right| = \sum_{i=1}^{n} a_i |y_i| + a \left| \sum_{v=1}^{n} y_v \right| - \sum_{s=n+2}^{N} b_s \left| \sum_{v=1}^{n} r_{s,v} y_v \right|$$

$$= \sum_{i=1}^{n} a_i |y_i| + a \left| \sum_{v=1}^{n} y_v \right| - \sum_{s \in S} b_s \left| \sum_{v=1}^{n} r_{s,v} y_v \right| \geq 0$$

by Theorem 13 (in fact, we were allowed to apply Theorem 13 because all the requirements of Theorem 13 are fulfilled - in particular, we have $a_i + a \geq \sum\limits_{s \in S} b_s r_{s,i}$ for every $i \in \{1, 2, ..., n\}$ because we know that $a_i + a = \sum\limits_{s \in S} b_s r_{s,i}$ for every $i \in \{1, 2, ..., n\}$ by an assumption of Theorem 14).

Altogether, we have now shown the following: The points z_1, z_2, ..., z_N are N points from I. The N reals u_1, u_2, ..., u_N satisfy $\sum\limits_{k=1}^{N} u_k = 0$, and $\sum\limits_{k=1}^{N} u_k |z_k - t| \geq 0$ holds for every $t \in \{z_1, z_2, ..., z_N\}$. Hence, according to Theorem 8b, we have $\sum\limits_{k=1}^{N} u_k f(z_k) \geq 0$. And as we have seen above, once $\sum\limits_{k=1}^{N} u_k f(z_k) \geq 0$ is shown, the proof of Theorem 14 is complete. Thus, Theorem 14 is proven.

7. Proving the Popoviciu inequality

Here is a very obvious lemma:

Theorem 15. Let N be a finite set, let m be an integer, and let i be an element of N. Then,

$$\sum_{s \subseteq N;\ |s|=m} \left\{ \begin{array}{l} 1, \text{ if } i \in s; \\ 0, \text{ if } i \notin s \end{array} \right. = \binom{|N|-1}{m-1}.$$

(Note that, though it may sound unusual, we refer to subsets of N by a minor letter s here and in the following.)

Proof of Theorem 15. The sum $\sum\limits_{s\subseteq N;\ |s|=m} \begin{cases} 1, & \text{if } i\in s; \\ 0, & \text{if } i\notin s \end{cases}$ equals to the number of all m-element subsets $s\subseteq N$ satisfying $i\in s$ (because every such subset s contributes a 1 to the sum, and all other subsets contribute 0's). But the number of all m-element subsets $s\subseteq N$ satisfying $i\in s$ is equal to $\binom{|N|-1}{m-1}$ (because such subsets are in a one-to-one correspondence with the $(m-1)$-element subsets $t\subseteq N\setminus\{i\}$ (this correspondence is given by $t=s\setminus\{i\}$ and, conversely, $s=t\cup\{i\}$), and the number of all $(m-1)$-element subsets $t\subseteq N\setminus\{i\}$ is $\binom{|N\setminus\{i\}|}{m-1}=\binom{|N|-1}{m-1}$). Thus, $\sum\limits_{s\subseteq N;\ |s|=m} \begin{cases} 1, & \text{if } i\in s; \\ 0, & \text{if } i\notin s \end{cases} = \binom{|N|-1}{m-1}$, and Theorem 15 is proven.

Now we can finally step to the *proof of Theorem 5b:*

We assume that $n\geq 2$, because all cases where $n<2$ (that is, $n=1$ or $n=0$) can be checked manually (and are uninteresting).

Let $a_i=\binom{n-2}{m-1}$ for every $i\in\{1,2,...,n\}$. Let $a=\binom{n-2}{m-2}$. These reals a_1, a_2, ..., a_n and a are all nonnegative (since $n\geq 2$ yields $n-2\geq 0$ and thus $\binom{n-2}{t}\geq 0$ for all integers t).

Let $S=\{s\subseteq\{1,2,...,n\}\mid |s|=m\}$; that is, we denote by S the set of all m-element subsets of the set $\{1,2,...,n\}$. This set S is obviously finite.

For every $s\in S$, define n reals $r_{s,1}$, $r_{s,2}$, ..., $r_{s,n}$ as follows:

$$r_{s,i}=\begin{cases} 1, & \text{if } i\in s; \\ 0, & \text{if } i\notin s \end{cases} \qquad \text{for every } i\in\{1,2,...,n\}.$$

Obviously, these reals $r_{s,1}$, $r_{s,2}$, ..., $r_{s,n}$ are all nonnegative. Also, for every $s\in S$, set $b_s=1$; then, b_s is a nonnegative real as well.

For every $i\in\{1,2,...,n\}$, we have

$$\begin{aligned}\sum_{s\in S} b_s r_{s,i} &= \sum_{s\in S} 1 r_{s,i} = \sum_{s\in S} r_{s,i} = \sum_{s\in S} \begin{cases} 1, & \text{if } i\in s; \\ 0, & \text{if } i\notin s \end{cases} = \sum_{\substack{s\subseteq\{1,2,...,n\};\\ |s|=m}} \begin{cases} 1, & \text{if } i\in s; \\ 0, & \text{if } i\notin s \end{cases} \\ &= \binom{|\{1,2,...,n\}|-1}{m-1} \qquad \text{(by Theorem 15 for } N=\{1,2,...,n\}) \\ &= \binom{n-1}{m-1},\end{aligned}$$

so that

$$\begin{aligned} a_i+a &= \binom{n-2}{m-1}+\binom{n-2}{m-2}=\binom{n-1}{m-1} \\ &\qquad \text{(by the recurrence relation of the binomial coefficients)} \\ &= \sum_{s\in S} b_s r_{s,i}. \end{aligned} \tag{11}$$

For any two distinct integers i and j from $\{1, 2, ..., n\}$, we have

$$\sum_{s\in S} b_s |r_{s,i} - r_{s,j}| = \sum_{s\in S} 1 |r_{s,i} - r_{s,j}| = \sum_{s\in S} |r_{s,i} - r_{s,j}|$$

$$= \sum_{s\in S} \left| \begin{cases} 1, & \text{if } i \in s; \\ 0, & \text{if } i \notin s \end{cases} - \begin{cases} 1, & \text{if } j \in s; \\ 0, & \text{if } j \notin s \end{cases} \right| = \sum_{s\in S} \left| \begin{cases} 1-1, & \text{if } i \in s \text{ and } j \in s; \\ 1-0, & \text{if } i \in s \text{ and } j \notin s; \\ 0-1, & \text{if } i \notin s \text{ and } j \in s; \\ 0-0, & \text{if } i \notin s \text{ and } j \notin s \end{cases} \right|$$

$$= \sum_{s\in S} \left| \begin{cases} 0, & \text{if } i \in s \text{ and } j \in s; \\ 1, & \text{if } i \in s \text{ and } j \notin s; \\ -1, & \text{if } i \notin s \text{ and } j \in s; \\ 0, & \text{if } i \notin s \text{ and } j \notin s \end{cases} \right| = \sum_{s\in S} \begin{cases} 0, & \text{if } i \in s \text{ and } j \in s; \\ 1, & \text{if } i \in s \text{ and } j \notin s; \\ 1, & \text{if } i \notin s \text{ and } j \in s; \\ 0, & \text{if } i \notin s \text{ and } j \notin s \end{cases}$$

$$= \sum_{s\in S} \left(\begin{cases} 1, & \text{if } i \in s \text{ and } j \notin s; \\ 0 \text{ otherwise} \end{cases} + \begin{cases} 1, & \text{if } i \notin s \text{ and } j \in s; \\ 0 \text{ otherwise} \end{cases} \right)$$

$$\left(\begin{array}{c} \text{because the cases } (i \in s \text{ and } j \notin s) \text{ and } (i \notin s \text{ and } j \in s) \\ \text{cannot occur simultaneously} \end{array} \right)$$

$$= \sum_{s\in S} \begin{cases} 1, & \text{if } i \in s \text{ and } j \notin s; \\ 0 \text{ otherwise} \end{cases} + \sum_{s\in S} \begin{cases} 1, & \text{if } i \notin s \text{ and } j \in s; \\ 0 \text{ otherwise} \end{cases}$$

$$= \sum_{s\in S} \begin{cases} 1, & \text{if } i \in s \text{ and } j \notin s; \\ 0 \text{ otherwise} \end{cases} + \sum_{s\in S} \begin{cases} 1, & \text{if } j \in s \text{ and } i \notin s; \\ 0 \text{ otherwise} \end{cases}.$$

Now,

$$\sum_{s\in S} \begin{cases} 1, & \text{if } i \in s \text{ and } j \notin s; \\ 0 \text{ otherwise} \end{cases} = \sum_{\substack{s\subseteq\{1,2,...,n\}; \\ |s|=m}} \begin{cases} 1, & \text{if } i \in s \text{ and } j \notin s; \\ 0 \text{ otherwise} \end{cases}$$

$$= \sum_{\substack{s\subseteq\{1,2,...,n\}; \\ |s|=m;\ j\notin s}} \begin{cases} 1, & \text{if } i \in s \text{ and } j \notin s; \\ 0 \text{ otherwise} \end{cases} \qquad \left(\begin{array}{c} \text{because all terms of the sum} \\ \text{with } j \in s \text{ are zero} \end{array} \right)$$

$$= \sum_{\substack{s\subseteq\{1,2,...,n\}; \\ |s|=m;\ j\notin s}} \begin{cases} 1, & \text{if } i \in s; \\ 0 \text{ otherwise} \end{cases} = \sum_{\substack{s\subseteq\{1,2,...,n\}\setminus\{j\}; \\ |s|=m}} \begin{cases} 1, & \text{if } i \in s; \\ 0 \text{ otherwise} \end{cases}$$

$$= \sum_{\substack{s\subseteq\{1,2,...,n\}\setminus\{j\}; \\ |s|=m}} \begin{cases} 1, & \text{if } i \in s; \\ 0, & \text{if } i \notin s \end{cases} = \binom{|\{1,2,...,n\} \setminus \{j\}| - 1}{m-1}$$

$$\left(\begin{array}{c} \text{by Theorem 15 for } N = \{1,2,...,n\} \setminus \{j\}; \text{ here, we use that } i \text{ is an} \\ \text{element of } N \text{ (because } i \in \{1,2,...,n\} \setminus \{j\}, \text{ since } i \text{ and } j \text{ are distinct)} \end{array} \right)$$

$$= \binom{(n-1)-1}{m-1} = \binom{n-2}{m-1} = a_i,$$

and similarly $\sum\limits_{s\in S}\left\{\begin{array}{c} 1,\text{ if } j\in s \text{ and } i\notin s;\\ 0 \text{ otherwise}\end{array}\right. = a_j$. Thus,

$$\begin{aligned} & a_i + a_j \\ &= \sum_{s\in S}\left\{\begin{array}{c} 1,\text{ if } i\in s \text{ and } j\notin s;\\ 0 \text{ otherwise}\end{array}\right. + \sum_{s\in S}\left\{\begin{array}{c} 1,\text{ if } j\in s \text{ and } i\notin s;\\ 0 \text{ otherwise}\end{array}\right. \\ &= \sum_{s\in S} b_s \left| r_{s,i} - r_{s,j}\right| . \end{aligned} \tag{12}$$

Also,

$$\sum_{v=1}^{n} w_v = w_1 + w_2 + ... + w_n \neq 0 \tag{13}$$

(by an assumption of Theorem 5b).

The elements of S are all the m-element subsets of $\{1,2,...,n\}$. Hence, to every element $s\in S$ uniquely correspond m integers i_1, i_2, ..., i_m satisfying $1\leq i_1 < i_2 < ... < i_m \leq n$ and $s=\{i_1,i_2,...,i_m\}$ (these m integers i_1, i_2, ..., i_m are the m elements of s in increasing order). And conversely, any m integers i_1, i_2, ..., i_m satisfying $1\leq i_1 < i_2 < ... < i_m \leq n$ can be obtained this way - in fact, they correspond to the m-element set $s=\{i_1,i_2,...,i_m\}\in S$. Given an element $s\in S$ and the corresponding m integers i_1, i_2, ..., i_m, we can write

$$\sum_{v=1}^{n} r_{s,v} w_v = \sum_{v=1}^{n}\left\{\begin{array}{c} 1,\text{ if } v\in s;\\ 0,\text{ if } v\notin s\end{array}\right. \cdot w_v = \sum_{v\in s} w_v = \sum_{v\in\{i_1,i_2,...,i_m\}} w_v = w_{i_1} + w_{i_2} + ... + w_{i_m};$$

$$\begin{aligned} \sum_{v=1}^{n} r_{s,v} w_v x_v &= \sum_{v=1}^{n}\left\{\begin{array}{c} 1,\text{ if } v\in s;\\ 0,\text{ if } v\notin s\end{array}\right. \cdot w_v x_v = \sum_{v\in s} w_v x_v \\ &= \sum_{v\in\{i_1,i_2,...,i_m\}} w_v x_v = w_{i_1}x_{i_1} + w_{i_2}x_{i_2} + ... + w_{i_m}x_{i_m}. \end{aligned}$$

From this, we can conclude that

$$\sum_{v=1}^{n} r_{s,v} w_v \neq 0 \qquad \text{for every } s\in S \tag{14}$$

(because $\sum\limits_{v=1}^{n} r_{s,v} w_v = w_{i_1} + w_{i_2} + ... + w_{i_m}$, and $w_{i_1} + w_{i_2} + ... + w_{i_m} \neq 0$ by an assumption of Theorem 5b), and we can also conclude that

$$\begin{aligned} & \sum_{s\in S}\left(\sum_{v=1}^{n} r_{s,v} w_v\right) f\left(\frac{\sum\limits_{v=1}^{n} r_{s,v} w_v x_v}{\sum\limits_{v=1}^{n} r_{s,v} w_v}\right) \\ &= \sum_{1\leq i_1 < i_2 < ... < i_m \leq n} \left(w_{i_1} + w_{i_2} + ... + w_{i_m}\right) f\left(\frac{w_{i_1}x_{i_1} + w_{i_2}x_{i_2} + ... + w_{i_m}x_{i_m}}{w_{i_1} + w_{i_2} + ... + w_{i_m}}\right). \end{aligned} \tag{15}$$

Using the conditions of Theorem 5b and the relations (11), (12), (13) and (14), we see that all conditions of Theorem 14 are fulfilled. Thus, we can apply Theorem 14, and obtain

$$\sum_{i=1}^{n} a_i w_i f\left(x_i\right) + a\left(\sum_{v=1}^{n} w_v\right) f\left(\frac{\sum\limits_{v=1}^{n} w_v x_v}{\sum\limits_{v=1}^{n} w_v}\right) \geq \sum_{s\in S} b_s \left(\sum_{v=1}^{n} r_{s,v} w_v\right) f\left(\frac{\sum\limits_{v=1}^{n} r_{s,v} w_v x_v}{\sum\limits_{v=1}^{n} r_{s,v} w_v}\right).$$

This rewrites as

$$\sum_{i=1}^{n} \binom{n-2}{m-1} w_i f\left(x_i\right) + \binom{n-2}{m-2} \left(\sum_{v=1}^{n} w_v\right) f\left(\frac{\sum\limits_{v=1}^{n} w_v x_v}{\sum\limits_{v=1}^{n} w_v}\right)$$
$$\geq \sum_{s\in S} 1 \left(\sum_{v=1}^{n} r_{s,v} w_v\right) f\left(\frac{\sum\limits_{v=1}^{n} r_{s,v} w_v x_v}{\sum\limits_{v=1}^{n} r_{s,v} w_v}\right).$$

In other words,

$$\binom{n-2}{m-1} \sum_{i=1}^{n} w_i f\left(x_i\right) + \binom{n-2}{m-2} \left(\sum_{v=1}^{n} w_v\right) f\left(\frac{\sum\limits_{v=1}^{n} w_v x_v}{\sum\limits_{v=1}^{n} w_v}\right)$$
$$\geq \sum_{s\in S} \left(\sum_{v=1}^{n} r_{s,v} w_v\right) f\left(\frac{\sum\limits_{v=1}^{n} r_{s,v} w_v x_v}{\sum\limits_{v=1}^{n} r_{s,v} w_v}\right).$$

Using (15) and the obvious relations

$$\sum_{v=1}^{n} w_v = w_1 + w_2 + ... + w_n;$$
$$\sum_{v=1}^{n} w_v x_v = w_1 x_1 + w_2 x_2 + ... + w_n x_n,$$

we can rewrite this as

$$\binom{n-2}{m-1} \sum_{i=1}^{n} w_i f\left(x_i\right) + \binom{n-2}{m-2} \left(w_1 + w_2 + ... + w_n\right) f\left(\frac{w_1 x_1 + w_2 x_2 + ... + w_n x_n}{w_1 + w_2 + ... + w_n}\right)$$
$$\geq \sum_{1\leq i_1 < i_2 < ... < i_m \leq n} \left(w_{i_1} + w_{i_2} + ... + w_{i_m}\right) f\left(\frac{w_{i_1} x_{i_1} + w_{i_2} x_{i_2} + ... + w_{i_m} x_{i_m}}{w_{i_1} + w_{i_2} + ... + w_{i_m}}\right).$$

This proves Theorem 5b.

8. A cyclic inequality

The most general form of the Popoviciu inequality is now proven. But this is not the end to the applications of Theorem 14. We will now apply it to show a cyclic inequality similar to Popoviciu's:

> **Theorem 16a.** Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Let x_1, x_2, ..., x_n be finitely many points from I.
>
> We extend the indices in x_1, x_2, ..., x_n cyclically modulo n; this means that for any integer $i \notin \{1, 2, ..., n\}$, we define a real x_i by setting $x_i = x_j$, where j is the integer from the set $\{1, 2, ..., n\}$ such that $i \equiv j \mod n$. (For instance, this means that $x_{n+3} = x_3$.)
>
> Let $x = \dfrac{x_1 + x_2 + ... + x_n}{n}$. Let r be an integer. Then,
>
> $$2\sum_{i=1}^{n} f(x_i) + n(n-2) f(x) \geq n \sum_{s=1}^{n} f\left(x + \frac{x_s - x_{s+r}}{n}\right).$$

A weighted version of this inequality is:

> **Theorem 16b.** Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Let x_1, x_2, ..., x_n be finitely many points from I. Let r be an integer.
>
> Let w_1, w_2, ..., w_n be nonnegative reals. Let $x = \dfrac{\sum\limits_{v=1}^{n} w_v x_v}{\sum\limits_{v=1}^{n} w_v}$ and $w = \sum\limits_{v=1}^{n} w_v$.
>
> Assume that $w \neq 0$ and that $w + (w_s - w_{s+r}) \neq 0$ for every $s \in S$.
>
> We extend the indices in x_1, x_2, ..., x_n and in w_1, w_2, ..., w_n cyclically modulo n; this means that for any integer $i \notin \{1, 2, ..., n\}$, we define reals x_i and w_i by setting $x_i = x_j$ and $w_i = w_j$, where j is the integer from the set $\{1, 2, ..., n\}$ such that $i \equiv j \mod n$. (For instance, this means that $x_{n+3} = x_3$ and $w_{n+2} = w_2$.)
>
> Then,
>
> $$2\sum_{i=1}^{n} w_i f(x_i) + (n-2) w f(x) \geq \sum_{s=1}^{n} (w + (w_s - w_{s+r})) f\left(\frac{\sum\limits_{v=1}^{n} w_v x_v + (w_s x_s - w_{s+r} x_{s+r})}{w + (w_s - w_{s+r})}\right).$$

Proof of Theorem 16b. We assume that $n \geq 2$, because all cases where $n < 2$ (that is, $n = 1$ or $n = 0$) can be checked manually (and are uninteresting).

Before we continue with the proof, let us introduce a simple notation: For any assertion $\mathcal{A}$, we denote by $[\mathcal{A}]$ the Boolean value of the assertion $\mathcal{A}$ (that is, $[\mathcal{A}] = \begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}$). Therefore, $0 \leq [\mathcal{A}] \leq 1$ for every assertion $\mathcal{A}$.

Let $a_i = 2$ for every $i \in \{1, 2, ..., n\}$. Let $a = n - 2$. These reals a_1, a_2, ..., a_n and a are all nonnegative (since $n \geq 2$ yields $n - 2 \geq 0$).

Let $S = \{1, 2, ..., n\}$. This set S is obviously finite.

For every $s \in S$, define n reals $r_{s,1}$, $r_{s,2}$, ..., $r_{s,n}$ as follows:

$$r_{s,i} = 1 + [i = s] - [i \equiv s + r \operatorname{mod} n] \qquad \text{for every } i \in \{1, 2, ..., n\}.$$

These reals $r_{s,1}$, $r_{s,2}$, ..., $r_{s,n}$ are all nonnegative (because

$$r_{s,i} = 1 + \underbrace{[i = s]}_{\geq 0} - \underbrace{[i \equiv s + r \operatorname{mod} n]}_{\leq 1} \geq 1 + 0 - 1 = 0$$

for every $i \in \{1, 2, ..., n\}$). Also, for every $s \in S$, set $b_s = 1$; then, b_s is a nonnegative real as well.

For every $i \in \{1, 2, ..., n\}$, we have

$$\sum_{s=1}^{n} [i = s] = \sum_{s=1}^{n} \begin{cases} 1, & \text{if } i = s; \\ 0 & \text{otherwise} \end{cases} = 1$$

(because there exists one and only one $s \in \{1, 2, ..., n\}$ satisfying $i = s$). Also, for every $i \in \{1, 2, ..., n\}$, we have

$$\sum_{s=1}^{n} [s \equiv i - r \operatorname{mod} n] = \sum_{s=1}^{n} \begin{cases} 1, & \text{if } s \equiv i - r \operatorname{mod} n; \\ & 0 \text{ otherwise} \end{cases} = 1$$

(because there exists one and only one $s \in \{1, 2, ..., n\}$ satisfying $s \equiv i - r \operatorname{mod} n$). In other words, $\sum\limits_{s=1}^{n} [i \equiv s + r \operatorname{mod} n] = 1$ (because $[s \equiv i - r \operatorname{mod} n] = [i \equiv s + r \operatorname{mod} n]$, since the two assertions $s \equiv i - r \operatorname{mod} n$ and $i \equiv s + r \operatorname{mod} n$ are equivalent).

For every $i \in \{1, 2, ..., n\}$, we have

$$\begin{aligned} \sum_{s \in S} b_s r_{s,i} &= \sum_{s=1}^{n} \underbrace{b_s}_{=1} r_{s,i} = \sum_{s=1}^{n} r_{s,i} = \sum_{s=1}^{n} (1 + [i = s] - [i \equiv s + r \operatorname{mod} n]) \\ &= \sum_{s=1}^{n} 1 + \sum_{s=1}^{n} [i = s] - \sum_{s=1}^{n} [i \equiv s + r \operatorname{mod} n] = n + 1 - 1 = n = 2 + (n - 2) = a_i + a, \end{aligned}$$

so that

$$a_i + a = \sum_{s \in S} b_s r_{s,i}. \tag{16}$$

For any two integers i and j from $\{1,2,...,n\}$, we have

$$\begin{aligned}
\sum_{s=1}^{n}|r_{s,i}-1| &= \sum_{s=1}^{n}|(1+[i=s]-[i\equiv s+r \bmod n])-1| \\
&= \sum_{s=1}^{n}|[i=s]+(-[i\equiv s+r \bmod n])| \\
&\leq \sum_{s=1}^{n}(|[i=s]|+|-[i\equiv s+r \bmod n]|) \\
&\qquad \left(\begin{array}{c}\text{since } |[i=s]+(-[i\equiv s+r \bmod n])| \leq |[i=s]|+|-[i\equiv s+r \bmod n]| \\ \text{by the triangle inequality}\end{array}\right) \\
&= \sum_{s=1}^{n}([i=s]+[i\equiv s+r \bmod n]) \\
&\qquad \left(\begin{array}{c}\text{because } [i=s] \text{ and } [i\equiv s+r \bmod n] \text{ are nonnegative, so that} \\ |[i=s]|=[i=s] \text{ and } |-[i\equiv s+r \bmod n]|=[i\equiv s+r \bmod n]\end{array}\right) \\
&= \sum_{s=1}^{n}[i=s]+\sum_{s=1}^{n}[i\equiv s+r \bmod n]=1+1=2
\end{aligned}$$

and similarly $\sum\limits_{s=1}^{n}|r_{s,j}-1|\leq 2$, so that

$$\begin{aligned}
\sum_{s\in S}b_s|r_{s,i}-r_{s,j}| &= \sum_{s=1}^{n}\underbrace{b_s}_{=1}|r_{s,i}-r_{s,j}| = \sum_{s=1}^{n}|r_{s,i}-r_{s,j}| \\
&= \sum_{s=1}^{n}|(r_{s,i}-1)+(1-r_{s,j})| \leq \sum_{s=1}^{n}(|r_{s,i}-1|+|1-r_{s,j}|) \\
&\qquad \left(\begin{array}{c}\text{because } |(r_{s,i}-1)+(1-r_{s,j})| \leq |r_{s,i}-1|+|1-r_{s,j}| \\ \text{by the triangle inequality}\end{array}\right) \\
&= \sum_{s=1}^{n}(|r_{s,i}-1|+|r_{s,j}-1|) = \sum_{s=1}^{n}|r_{s,i}-1|+\sum_{s=1}^{n}|r_{s,j}-1| \\
&\leq 2+2=a_i+a_j,
\end{aligned}$$

and thus

$$a_i+a_j \geq \sum_{s\in S}b_s|r_{s,i}-r_{s,j}|. \tag{17}$$

For every $s\in S$ (that is, for every $s\in\{1,2,...,n\}$), we have

$$\begin{aligned}
\sum_{v=1}^{n}[v\equiv s+r \bmod n]\cdot w_v &= \sum_{v=1}^{n}\left\{\begin{array}{c}1, \text{ if } v\equiv s+r \bmod n; \\ 0 \text{ otherwise}\end{array}\right. \cdot w_v \\
&= \sum_{v=1}^{n}\left\{\begin{array}{c}w_v, \text{ if } v\equiv s+r \bmod n; \\ 0 \text{ otherwise}\end{array}\right. = w_{s+r}
\end{aligned}$$

(because there is one and only one element $v \in \{1, 2, ..., n\}$ that satisfies $v \equiv s + r \bmod n$, and for this element v, we have $w_v = w_{s+r}$), so that

$$\sum_{v=1}^{n} r_{s,v} w_v = \sum_{v=1}^{n} (1 + [v = s] - [v \equiv s + r \bmod n]) \cdot w_v$$
$$= \underbrace{\sum_{v=1}^{n} w_v}_{=w} + \underbrace{\sum_{v=1}^{n} [v = s] \cdot w_v}_{=w_s} - \underbrace{\sum_{v=1}^{n} [v \equiv s + r \bmod n] \cdot w_v}_{=w_{s+r}}$$
$$= w + w_s - w_{s+r} = w + (w_s - w_{s+r}).$$

Also, for every $s \in S$ (that is, for every $s \in \{1, 2, ..., n\}$), we have

$$\sum_{v=1}^{n} [v \equiv s + r \bmod n] \cdot w_v x_v = \sum_{v=1}^{n} \left\{ \begin{array}{c} 1, \text{ if } v \equiv s + r \bmod n; \\ 0 \text{ otherwise} \end{array} \right. \cdot w_v x_v$$
$$= \sum_{v=1}^{n} \left\{ \begin{array}{c} w_v x_v, \text{ if } v \equiv s + r \bmod n; \\ 0 \text{ otherwise} \end{array} \right. = w_{s+r} x_{s+r}$$

(because there is one and only one element $v \in \{1, 2, ..., n\}$ that satisfies $v \equiv s + r \bmod n$, and for this element v, we have $w_v = w_{s+r}$ and $x_v = x_{s+r}$), and thus

$$\sum_{v=1}^{n} r_{s,v} w_v x_v = \sum_{v=1}^{n} (1 + [v = s] - [v \equiv s + r \bmod n]) \cdot w_v x_v$$
$$= \sum_{v=1}^{n} w_v x_v + \underbrace{\sum_{v=1}^{n} [v = s] \cdot w_v x_v}_{=w_s x_s} - \underbrace{\sum_{v=1}^{n} [v \equiv s + r \bmod n] \cdot w_v x_v}_{=w_{s+r} x_{s+r}}$$
$$= \sum_{v=1}^{n} w_v x_v + w_s x_s - w_{s+r} x_{s+r} = \sum_{v=1}^{n} w_v x_v + (w_s x_s - w_{s+r} x_{s+r}).$$

Now it is clear that $\sum\limits_{v=1}^{n} r_{s,v} w_v \neq 0$ for all $s \in S$ (because $\sum\limits_{v=1}^{n} r_{s,v} w_v = w + (w_s - w_{s+r})$ and $w + (w_s - w_{s+r}) \neq 0$). Also, $\sum\limits_{v=1}^{n} w_v \neq 0$ (since $\sum\limits_{v=1}^{n} w_v = w$ and $w \neq 0$). Using these two relations, the conditions of Theorem 16b and the relations (16) and (17), we see that all conditions of Theorem 14 are fulfilled. Hence, we can apply Theorem 14 and obtain

$$\sum_{i=1}^{n} \underbrace{a_i}_{=2} w_i f(x_i) + \underbrace{a}_{=n-2} \left(\underbrace{\sum_{v=1}^{n} w_v}_{=w} \right) f \left(\underbrace{\frac{\sum\limits_{v=1}^{n} w_v x_v}{\sum\limits_{v=1}^{n} w_v}}_{=x} \right) \geq \sum_{s \in S} \underbrace{b_s}_{=1} \left(\sum_{v=1}^{n} r_{s,v} w_v \right) f \left(\frac{\sum\limits_{v=1}^{n} r_{s,v} w_v x_v}{\sum\limits_{v=1}^{n} r_{s,v} w_v} \right).$$

This immediately simplifies to

$$\sum_{i=1}^{n} 2 w_i f(x_i) + (n-2) w f(x) \geq \sum_{s \in S} 1 \left(\sum_{v=1}^{n} r_{s,v} w_v \right) f \left(\frac{\sum\limits_{v=1}^{n} r_{s,v} w_v x_v}{\sum\limits_{v=1}^{n} r_{s,v} w_v} \right).$$

Recalling that for every $s \in S$, we have $\sum\limits_{v=1}^{n} r_{s,v} w_v = w + (w_s - w_{s+r})$ and $\sum\limits_{v=1}^{n} r_{s,v} w_v x_v = \sum\limits_{v=1}^{n} w_v x_v + (w_s x_s - w_{s+r} x_{s+r})$, we can rewrite this as

$$\sum_{i=1}^{n} 2 w_i f(x_i) + (n-2) w f(x) \geq \sum_{s \in S} 1 (w + (w_s - w_{s+r})) f\left(\frac{\sum\limits_{v=1}^{n} w_v x_v + (w_s x_s - w_{s+r} x_{s+r})}{w + (w_s - w_{s+r})} \right).$$

In other words,

$$2 \sum_{i=1}^{n} w_i f(x_i) + (n-2) w f(x) \geq \sum_{s \in S} (w + (w_s - w_{s+r})) f\left(\frac{\sum\limits_{v=1}^{n} w_v x_v + (w_s x_s - w_{s+r} x_{s+r})}{w + (w_s - w_{s+r})} \right).$$

Equivalently,

$$2 \sum_{i=1}^{n} w_i f(x_i) + (n-2) w f(x) \geq \sum_{s=1}^{n} (w + (w_s - w_{s+r})) f\left(\frac{\sum\limits_{v=1}^{n} w_v x_v + (w_s x_s - w_{s+r} x_{s+r})}{w + (w_s - w_{s+r})} \right).$$

This proves Theorem 16b.

Proof of Theorem 16a. Define n reals $w_1, w_2, ..., w_n$ by setting $w_i = 1$ for every $i \in \{1, 2, ..., n\}$. Obviously, these reals $w_1, w_2, ..., w_n$ are nonnegative.

Define $w = \sum\limits_{v=1}^{n} w_v$. Then, $w = \sum\limits_{v=1}^{n} w_v = \sum\limits_{v=1}^{n} 1 = n \cdot 1 = n$. Also,

$$x = \frac{x_1 + x_2 + ... + x_n}{n} = \frac{\sum\limits_{v=1}^{n} x_v}{n} = \frac{\sum\limits_{v=1}^{n} 1 x_v}{n} = \frac{\sum\limits_{v=1}^{n} 1 x_v}{w} = \frac{\sum\limits_{v=1}^{n} w_v x_v}{\sum\limits_{v=1}^{n} w_v}$$

(since $1 = w_v$ and $w = \sum\limits_{v=1}^{n} w_v$). Also, $w \neq 0$ (since $w = n$) and $w + (w_s - w_{s+r}) \neq 0$ for every $s \in S$ (since $w + (w_s - w_{s+r}) = n + (1 - 1) = n$).

We summarize: The n nonnegative reals $w_1, w_2, ..., w_n$ and the reals $w = \sum\limits_{v=1}^{n} w_v$ and $x = \frac{\sum\limits_{v=1}^{n} w_v x_v}{\sum\limits_{v=1}^{n} w_v}$ satisfy $w \neq 0$ and $w + (w_s - w_{s+r}) \neq 0$ for every $s \in S$. Therefore, all conditions of Theorem 16b are fulfilled. Hence, we can apply Theorem 16b and obtain

$$2 \sum_{i=1}^{n} w_i f(x_i) + (n-2) w f(x) \geq \sum_{s=1}^{n} (w + (w_s - w_{s+r})) f\left(\frac{\sum\limits_{v=1}^{n} w_v x_v + (w_s x_s - w_{s+r} x_{s+r})}{w + (w_s - w_{s+r})} \right).$$

Since $w_i = 1$ for all $i \in \{1, 2, ..., n\}$ and $w = n$, this rewrites as

$$2 \sum_{i=1}^{n} 1 f(x_i) + (n-2) n f(x) \geq \sum_{s=1}^{n} (n + (1-1)) f\left(\frac{\sum_{v=1}^{n} 1 x_v + (1 x_s - 1 x_{s+r})}{n + (1-1)}\right).$$

Since

$$\sum_{s=1}^{n} (n + (1-1)) f\left(\frac{\sum_{v=1}^{n} 1 x_v + (1 x_s - 1 x_{s+r})}{n + (1-1)}\right) = \sum_{s=1}^{n} n f\left(\frac{\sum_{v=1}^{n} x_v + (x_s - x_{s+r})}{n}\right)$$

$$= \sum_{s=1}^{n} n f\left(\frac{\sum_{v=1}^{n} x_v}{n} + \frac{x_s - x_{s+r}}{n}\right) = \sum_{s=1}^{n} n f\left(\frac{x_1 + x_2 + ... + x_n}{n} + \frac{x_s - x_{s+r}}{n}\right)$$

$$= \sum_{s=1}^{n} n f\left(x + \frac{x_s - x_{s+r}}{n}\right) = n \sum_{s=1}^{n} f\left(x + \frac{x_s - x_{s+r}}{n}\right),$$

this becomes

$$2 \sum_{i=1}^{n} 1 f(x_i) + (n-2) n f(x) \geq n \sum_{s=1}^{n} f\left(x + \frac{x_s - x_{s+r}}{n}\right).$$

In other words,

$$2 \sum_{i=1}^{n} f(x_i) + n(n-2) f(x) \geq n \sum_{s=1}^{n} f\left(x + \frac{x_s - x_{s+r}}{n}\right).$$

Thus, Theorem 16a is proven.

9. Applications of Theorem 16a

Finally we are going to show two easy applications of the above Theorem 16a. First, if we apply Theorem 16a to $r = 1$, to $r = 2$, to $r = 3$, and so on up to $r = n - 1$, and sum up the $n - 1$ inequalities obtained, then we get:

Theorem 17. Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Let x_1, x_2, ..., x_n be finitely many points from I.
Let $x = \dfrac{x_1 + x_2 + ... + x_n}{n}$. Then,

$$2(n-1) \sum_{i=1}^{n} f(x_i) + n(n-1)(n-2) f(x) \geq n \sum_{\substack{1 \leq i \leq n; \\ 1 \leq j \leq n; \\ i \neq j}} f\left(x + \frac{x_i - x_j}{n}\right).$$

The details of deducing this inequality from Theorem 16a are left to the reader. I am only mentioning Theorem 17 because it occured in [6], post #4 as a result by Vasile Cîrtoaje (Vasc). Our Theorem 16a is therefore a strengthening of this result.

The next theorem is just a rewritten particular case of Theorem 16a:

Theorem 18. Let f be a convex function from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$. Let A, B, C, D be four points from I. Then,

$$f(A)+f(B)+f(C)+f(D)+4f\left(\frac{A+B+C+D}{4}\right)$$
$$\geq 2\left(f\left(\frac{2A+B+C}{4}\right)+f\left(\frac{2B+C+D}{4}\right)+f\left(\frac{2C+D+A}{4}\right)+f\left(\frac{2D+A+B}{4}\right)\right).$$

Proof of Theorem 18. Set $x_1 = A$, $x_2 = B$, $x_3 = C$, $x_4 = D$. Then, x_1, x_2, x_3, x_4 are finitely many (namely, four) points from I (because A, B, C, D are four points from I).

We extend the indices in x_1, x_2, x_3, x_4 cyclically modulo 4; this means that for any integer $i \notin \{1,2,3,4\}$, we define a real x_i by setting $x_i = x_j$, where j is the integer from the set $\{1,2,3,4\}$ such that $i \equiv j \bmod 4$. (For instance, this means that $x_6 = x_2$.)

Let $x = \dfrac{x_1+x_2+x_3+x_4}{4}$. Then, we can apply Theorem 16a with $n = 4$ and $r = 3$, and we obtain

$$2\sum_{i=1}^{n} f(x_i) + n(n-2)f(x) \geq n\sum_{s=1}^{n} f\left(x+\frac{x_s-x_{s+r}}{n}\right),$$

where $n = 4$ and $r = 3$. In other words,

$$2\sum_{i=1}^{4} f(x_i) + 4(4-2)f(x) \geq 4\sum_{s=1}^{4} f\left(x+\frac{x_s-x_{s+3}}{4}\right). \tag{18}$$

Since $x_1 = A$, $x_2 = B$, $x_3 = C$, $x_4 = D$, $x_5 = x_1 = A$, $x_6 = x_2 = B$, $x_7 = x_3 = C$ and $x = \dfrac{x_1+x_2+x_3+x_4}{4} = \dfrac{A+B+C+D}{4}$, we have

$$\sum_{i=1}^{4} f(x_i) = f(x_1)+f(x_2)+f(x_3)+f(x_4) = f(A)+f(B)+f(C)+f(D);$$

$$4(4-2)f(x) = 4\cdot 2\cdot f\left(\frac{A+B+C+D}{4}\right);$$

$$\sum_{s=1}^{4} f\left(x+\frac{x_s-x_{s+3}}{4}\right)$$
$$= f\left(x+\frac{x_1-x_4}{4}\right)+f\left(x+\frac{x_2-x_5}{4}\right)+f\left(x+\frac{x_3-x_6}{4}\right)+f\left(x+\frac{x_4-x_7}{4}\right)$$
$$= f\left(\frac{A+B+C+D}{4}+\frac{A-D}{4}\right)+f\left(\frac{A+B+C+D}{4}+\frac{B-A}{4}\right)$$
$$+ f\left(\frac{A+B+C+D}{4}+\frac{C-B}{4}\right)+f\left(\frac{A+B+C+D}{4}+\frac{D-C}{4}\right)$$
$$= f\left(\frac{2A+B+C}{4}\right)+f\left(\frac{2B+C+D}{4}\right)+f\left(\frac{2C+D+A}{4}\right)+f\left(\frac{2D+A+B}{4}\right),$$

and thus (18) becomes

$$2\left(f\left(A\right)+f\left(B\right)+f\left(C\right)+f\left(D\right)\right)+4\cdot 2\cdot f\left(\frac{A+B+C+D}{4}\right)$$
$$\geq 4\left(f\left(\frac{2A+B+C}{4}\right)+f\left(\frac{2B+C+D}{4}\right)+f\left(\frac{2C+D+A}{4}\right)+f\left(\frac{2D+A+B}{4}\right)\right).$$

Dividing this inequality by 2, we obtain

$$f\left(A\right)+f\left(B\right)+f\left(C\right)+f\left(D\right)+4f\left(\frac{A+B+C+D}{4}\right)$$
$$\geq 2\left(f\left(\frac{2A+B+C}{4}\right)+f\left(\frac{2B+C+D}{4}\right)+f\left(\frac{2C+D+A}{4}\right)+f\left(\frac{2D+A+B}{4}\right)\right).$$

Thus, Theorem 18 is proven.

We will use this Theorem 18 to prove an inequality from Michael Rozenberg (aka "Arqady") in [7]:

Theorem 19. Let a, b, c, d be four nonnegative reals. Then,

$$a^4+b^4+c^4+d^4+4abcd\geq 2\left(a^2bc+b^2cd+c^2da+d^2ab\right).$$

Proof of Theorem 19. The case when at least one of the reals a, b, c, d equals 0 is easy (in fact, in this case, we can WLOG assume that $a=0$; then, the inequality in question,

$$a^4+b^4+c^4+d^4+4abcd\geq 2\left(a^2bc+b^2cd+c^2da+d^2ab\right)$$

is true because

$$\begin{aligned}&\left(a^4+b^4+c^4+d^4+4abcd\right)-2\left(a^2bc+b^2cd+c^2da+d^2ab\right)\\&=\left(0^4+b^4+c^4+d^4+4\cdot 0\cdot bcd\right)-2\left(0^2bc+b^2cd+c^2d\cdot 0+d^2\cdot 0b\right)\\&=\left(b^4+c^4+d^4\right)-2b^2cd=\underbrace{\left(b^2-cd\right)^2}_{\geq 0}+\underbrace{\left(c^2-d^2\right)^2}_{\geq 0}+\underbrace{c^2d^2}_{\geq 0}\geq 0\end{aligned}$$

). Hence, we can assume for the rest of this proof that none of the reals a, b, c, d equals 0. Since the reals a, b, c, d are nonnegative, this means that the reals a, b, c, d are positive.

Let $A=\ln\left(a^4\right)$, $B=\ln\left(b^4\right)$, $C=\ln\left(c^4\right)$, $D=\ln\left(d^4\right)$. Then, $\exp A=a^4$, $\exp B=b^4$, $\exp C=c^4$, $\exp D=d^4$.

Let $I\subseteq\mathbb{R}$ be an interval containing the reals A, B, C, D (for instance, $I=\mathbb{R}$). Let $f:I\to\mathbb{R}$ be the function defined by $f\left(x\right)=\exp x$ for all $x\in I$. Then, it is known that this function f is convex. Thus, Theorem 18 yields

$$f\left(A\right)+f\left(B\right)+f\left(C\right)+f\left(D\right)+4f\left(\frac{A+B+C+D}{4}\right)$$
$$\geq 2\left(f\left(\frac{2A+B+C}{4}\right)+f\left(\frac{2B+C+D}{4}\right)+f\left(\frac{2C+D+A}{4}\right)+f\left(\frac{2D+A+B}{4}\right)\right).$$

Since we have

$$\begin{aligned}
f(A) &= \exp A = a^4 \qquad \text{and similarly} \\
f(B) &= b^4,\ f(C) = c^4,\ \text{and } f(D) = d^4; \\
f\left(\frac{A+B+C+D}{4}\right) &= \exp\frac{A+B+C+D}{4} = \sqrt[4]{\exp A\cdot\exp B\cdot\exp C\cdot\exp D} \\
&= \sqrt[4]{a^4\cdot b^4\cdot c^4\cdot d^4} = abcd; \\
f\left(\frac{2A+B+C}{4}\right) &= \exp\frac{2A+B+C}{4} = \sqrt[4]{(\exp A)^2\cdot\exp B\cdot\exp C} \\
&= \sqrt[4]{(a^4)^2\cdot b^4\cdot c^4} = a^2bc \qquad \text{and similarly} \\
f\left(\frac{2B+C+D}{4}\right) &= b^2cd,\ f\left(\frac{2C+D+A}{4}\right) = c^2da,\ \text{and } f\left(\frac{2D+A+B}{4}\right) = d^2ab,
\end{aligned}$$

this becomes

$$a^4+b^4+c^4+d^4+4abcd \geq 2\left(a^2bc+b^2cd+c^2da+d^2ab\right).$$

This proves Theorem 19.

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