Mathematical Reflections
Problem O114 by Gabriel Dospinescu

Prove that for all real numbers x, y, z, the following inequality holds:

$$\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)\left(x^2+xy+y^2\right) \geq 3\left(x^2y+y^2z+z^2x\right)\left(xy^2+yz^2+zx^2\right).$$

Solution by Darij Grinberg.

The inequality in question immediately follows from the identity

$$\begin{aligned}&\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)\left(x^2+xy+y^2\right)\\&=3\left(x^2y+y^2z+z^2x\right)\left(xy^2+yz^2+zx^2\right)+\left(\left(x-y\right)\left(y-z\right)\left(z-x\right)\right)^2.\end{aligned}$$

What remains is to prove this identity. Of course, we can prove it by expanding, but here is a more conceptual proof:

Denote $a=x^2y+y^2z+z^2x$ and $b=xy^2+yz^2+zx^2$.

We work in $\mathbb{C}$. Let $\zeta=\dfrac{1+\sqrt{3}i}{2}$. Then, $\zeta^3=-1$ and thus

$$\begin{aligned}&\left(x+\zeta y\right)\left(y+\zeta z\right)\left(z+\zeta x\right)\\&=\left(\underbrace{\zeta^3+1}_{=-1+1=0}\right)xyz+\zeta\left(\underbrace{x^2y+y^2z+z^2x}_{=a}\right)+\zeta^2\underbrace{\left(xy^2+yz^2+zx^2\right)}_{=b}=\zeta a+\zeta^2 b\\&=\zeta\left(a+\zeta b\right).\end{aligned}$$

The same computation with ζ replaced by $\dfrac{1}{\zeta}$ everywhere (and using $\left(\dfrac{1}{\zeta}\right)^3=-1$ instead of $\zeta^3=-1$) proves

$$\left(x+\frac{1}{\zeta}y\right)\left(y+\frac{1}{\zeta}z\right)\left(z+\frac{1}{\zeta}x\right)=\frac{1}{\zeta}\left(a+\frac{1}{\zeta}b\right).$$

But any two complex numbers u and v satisfy

$$\left(u+\zeta v\right)\left(u+\frac{1}{\zeta}v\right)=u^2+uv+v^2 \qquad (1)$$

(since $\left(u+\zeta v\right)\left(u+\dfrac{1}{\zeta}v\right)=u^2+\left(\zeta+\dfrac{1}{\zeta}\right)uv+v^2$ and $\zeta+\dfrac{1}{\zeta}=1$ as we can easily see).

Hence,

$$\begin{aligned}
&\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)\left(x^2+xy+y^2\right)\\
&=(y+\zeta z)\left(y+\frac{1}{\zeta}z\right)(z+\zeta x)\left(z+\frac{1}{\zeta}x\right)(x+\zeta y)\left(x+\frac{1}{\zeta}y\right)\\
&\qquad\left(\begin{array}{c}\text{since (2) yields }(y+\zeta z)\left(y+\frac{1}{\zeta}z\right)=y^2+yz+z^2,\\ (z+\zeta x)\left(z+\frac{1}{\zeta}x\right)=z^2+zx+x^2\text{ and }(x+\zeta y)\left(x+\frac{1}{\zeta}y\right)=x^2+xy+y^2\end{array}\right)\\
&=(x+\zeta y)(y+\zeta z)(z+\zeta x)\cdot\left(x+\frac{1}{\zeta}y\right)\left(y+\frac{1}{\zeta}z\right)\left(z+\frac{1}{\zeta}x\right)\\
&=\zeta(a+\zeta b)\cdot\frac{1}{\zeta}\left(a+\frac{1}{\zeta}b\right)=(a+\zeta b)\left(a+\frac{1}{\zeta}b\right)=a^2+ab+b^2\qquad\text{(by (2))}\\
&=3ab+(b-a)^2\\
&=3\left(x^2y+y^2z+z^2x\right)\left(xy^2+yz^2+zx^2\right)+((x-y)(y-z)(z-x))^2
\end{aligned}$$

(since $a=x^2y+y^2z+z^2x$, $b=xy^2+yz^2+zx^2$, and a quick computation shows that

$$b-a=\left(xy^2+yz^2+zx^2\right)-\left(x^2y+y^2z+z^2x\right)=(x-y)(y-z)(z-x)$$

), qed.