Mathematical Reflections
Problem O112 by Cezar Lupu and Pham Huu Duc

Let a, b, c be positive real numbers. Prove that

$$\frac{a^3+abc}{(b+c)^2}+\frac{b^3+abc}{(c+a)^2}+\frac{c^3+abc}{(a+b)^2}\geq\frac{3}{2}\cdot\frac{a^3+b^3+c^3}{a^2+b^2+c^2}.$$

Solution by Darij Grinberg. We will use the $\sum$ sign to denote cyclic summation (i. e., summation over the cyclic transpositions of our 3 variables a, b, c).

WLOG assume that $a\geq b\geq c$ (we can assume this since the inequality which we must prove is symmetric). Set $x=\frac{a}{(b+c)^2}$, $y=\frac{b}{(c+a)^2}$, $z=\frac{c}{(a+b)^2}$. Then, $x\geq y$ [1]. Thus, the Vornicu-Schur inequality[2] yields

$$x(a-b)(a-c)+y(b-c)(b-a)+z(c-a)(c-b)\geq 0.$$

In other words,

$$\frac{a}{(b+c)^2}(a-b)(a-c)+\frac{b}{(c+a)^2}(b-c)(b-a)+\frac{c}{(a+b)^2}(c-a)(c-b)\geq 0.$$

Using the $\sum$ sign, this rewrites as

$$\sum\frac{a}{(b+c)^2}(a-b)(a-c)\geq 0. \tag{1}$$

Also, it is easily seen that

$$a+b+c=\frac{(a^3+b^3+c^3-2abc)+(b+c)(c+a)(a+b)}{a^2+b^2+c^2}. \tag{2}$$

[3]

Besides, Nesbitt's inequality states that

$$\sum\frac{a}{b+c}\geq\frac{3}{2}. \tag{3}$$

Finally,

$$a^3+b^3+c^3-2abc\geq 0 \tag{4}$$

[1] In fact, $a\geq b$ yields $c+a\geq b+c$, so that $(c+a)^2\geq(b+c)^2$, so that $\frac{1}{(b+c)^2}\geq\frac{1}{(c+a)^2}$; by multiplying this inequality with $a\geq b$ we obtain $\frac{a}{(b+c)^2}\geq\frac{b}{(c+a)^2}$; in other words, $x\geq y$.

[2] The "Vornicu-Schur inequality" that we use here is the following fact:

Let a, b, c be three reals, and let x, y, z be three nonnegative reals. If $a\geq b\geq c$ and $x\geq y$, then

$$x(a-b)(a-c)+y(b-c)(b-a)+z(c-a)(c-b)\geq 0.$$

This is Theorem 1 **a)** in [1]. The proof is fairly easy (just show that $x(a-b)(a-c)+y(b-c)(b-a)\geq 0$ and $z(c-a)(c-b)\geq 0$).

[3] I don't know how to prove (2) completely without computations, but here are two very short computational proofs of (2):

(since the AM-GM inequality yields $a^3+b^3+c^3 \geq 3\sqrt[3]{a^3b^3c^3} = 3abc \geq 2abc$).

Now,

$$\frac{a^3+abc}{(b+c)^2}+\frac{b^3+abc}{(c+a)^2}+\frac{c^3+abc}{(a+b)^2} = \sum \frac{a^3+abc}{(b+c)^2} = \sum \frac{a}{(b+c)^2}\left(a^2+bc\right)$$
$$= \sum \frac{a}{(b+c)^2}\left(a\left(b+c\right)+\left(a-b\right)\left(a-c\right)\right)$$
$$= \sum \frac{a}{(b+c)^2}a\left(b+c\right) + \underbrace{\sum \frac{a}{(b+c)^2}\left(a-b\right)\left(a-c\right)}_{\geq 0 \text{ by } (1)}$$
$$\geq \sum \frac{a}{(b+c)^2}a\left(b+c\right) = \sum \frac{a^2}{b+c} = \sum \left(\left(\frac{a^2}{b+c}+a\right)-a\right)$$
$$= \sum \left(\frac{a^2}{b+c}+a\right) - \sum a = \sum \frac{a^2+a\left(b+c\right)}{b+c} - \sum a = \sum \frac{a\left(a+b+c\right)}{b+c} - \sum a$$
$$= \left(a+b+c\right)\sum \frac{a}{b+c} - \sum a$$
$$= \frac{\left(a^3+b^3+c^3-2abc\right)+\left(b+c\right)\left(c+a\right)\left(a+b\right)}{a^2+b^2+c^2}\sum \frac{a}{b+c} - \sum a \qquad \text{(by (2))}$$
$$= \frac{1}{a^2+b^2+c^2}\left(\left(\left(a^3+b^3+c^3-2abc\right)+\left(b+c\right)\left(c+a\right)\left(a+b\right)\right)\sum \frac{a}{b+c} - \left(a^2+b^2+c^2\right)\sum a\right)$$

First proof of (2). The equality (2) follows from

$$\begin{aligned}\left(a+b+c\right)\left(a^2+b^2+c^2\right) &= a^3+b^3+c^3+b^2c+c^2b+c^2a+a^2c+a^2b+b^2a \\ &= \left(a^3+b^3+c^3-2abc\right)+\underbrace{\left(b^2c+c^2b+c^2a+a^2c+a^2b+b^2a+2abc\right)}_{=(b+c)(c+a)(a+b)} \\ &= \left(a^3+b^3+c^3-2abc\right)+\left(b+c\right)\left(c+a\right)\left(a+b\right).\end{aligned}$$

Second proof of (2). The equality (2) follows from

$$\begin{aligned}\left(a+b+c\right)\left(a^2+b^2+c^2\right) &= a\underbrace{\left(a^2+b^2+c^2\right)}_{=a^2-2bc+(b+c)^2}+\left(b+c\right)\left(a^2+b^2+c^2\right) \\ &= a^3-2abc+a\left(b+c\right)^2+\left(b+c\right)\left(a^2+b^2+c^2\right) \\ &= \left(a^3+b^3+c^3-2abc\right)+a\left(b+c\right)^2+\left(b+c\right)\left(a^2+b^2+c^2\right)-\left(b^3+c^3\right) \\ &= \left(a^3+b^3+c^3-2abc\right)+a\left(b+c\right)^2+\left(b+c\right)\left(a^2+b^2+c^2\right)-\left(b+c\right)\left(b^2-bc+c^2\right) \\ &= \left(a^3+b^3+c^3-2abc\right)+\left(b+c\right)\left(\underbrace{a\left(b+c\right)+\left(a^2+b^2+c^2\right)-\left(b^2-bc+c^2\right)}_{=a(b+c)+a^2+bc=(c+a)(a+b)}\right) \\ &= \left(a^3+b^3+c^3-2abc\right)+\left(b+c\right)\left(c+a\right)\left(a+b\right).\end{aligned}$$

$$= \frac{1}{a^2+b^2+c^2}\left(\underbrace{\left(a^3+b^3+c^3-2abc\right)\sum\frac{a}{b+c}}_{\substack{\geq\left(a^3+b^3+c^3-2abc\right)\cdot\frac{3}{2}\text{, since}\\ a^3+b^3+c^3-2abc\geq 0\text{ by (4) and}\\ \sum\frac{a}{b+c}\geq\frac{3}{2}\text{ by (3)}}} + \underbrace{(b+c)(c+a)(a+b)\sum\frac{a}{b+c}}_{\substack{=\sum a(c+a)(a+b)=\sum a(bc+a(a+b+c))\\ =\sum\left(abc+a^2(a+b+c)\right)\\ =\sum abc+(a+b+c)\sum a^2=3abc+\sum a\sum a^2}} - \underbrace{\left(a^2+b^2+c^2\right)\sum a}_{=\sum a^2\sum a=\sum a\sum a^2}\right)$$

$$\geq \frac{1}{a^2+b^2+c^2}\left(\left(a^3+b^3+c^3-2abc\right)\cdot\frac{3}{2}+\left(3abc+\sum a\sum a^2\right)-\sum a\sum a^2\right)$$

$$= \frac{1}{a^2+b^2+c^2}\left(\left(a^3+b^3+c^3-2abc\right)\cdot\frac{3}{2}+3abc\right) = \frac{1}{a^2+b^2+c^2}\left(\left(a^3+b^3+c^3\right)\cdot\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{a^3+b^3+c^3}{a^2+b^2+c^2},$$

qed.

References

[1] Darij Grinberg, *The Vornicu-Schur inequality and its variations*, version 13 August 2007.
http://de.geocities.com/darij_grinberg/VornicuS.pdf