Mathematical Reflections
Problem O111 by Titu Andreescu

Prove that, for each integer $n \geq 0$, the number

$$\left(\binom{n}{0}+2\binom{n}{2}+2^2\binom{n}{4}+...\right)^2\left(\binom{n}{1}+2\binom{n}{3}+2^2\binom{n}{5}+...\right)^2$$

is triangular.

Solution by Darij Grinberg. First, we consider a more general setting:

Theorem 1. Let q be a real. Define a sequence $(f_0, f_1, f_2, ...)$ by $f_n = \sum\limits_{k\in\mathbb{N}} \binom{n}{2k} q^k$ for every $n \in \mathbb{N}$, and define a sequence $(g_0, g_1, g_2, ...)$ by $g_n = \sum\limits_{k\in\mathbb{N}} \binom{n}{2k+1} q^k$ for every $n \in \mathbb{N}$. [1] Then, for every $n \in \mathbb{N}$, we have the matrix equality

$$\begin{pmatrix} f_n & qg_n \\ g_n & f_n \end{pmatrix} = \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^n \tag{1}$$

and the equalities

$$f_n^2 - qg_n^2 = (1-q)^n; \tag{2}$$

$$2f_ng_n = g_{2n}. \tag{3}$$

For any $a \in \mathbb{N}$ and $b \in \mathbb{N}$, we have

$$f_{a+b} = f_af_b + qg_ag_b; \tag{4}$$

$$g_{a+b} = f_ag_b + g_af_b. \tag{5}$$

Remark. Here, $\mathbb{N}$ means the set $\{0, 1, 2, ...\}$.
Proof of Theorem 1. We will prove (1) by induction:
Induction base. We have

$$f_0 = \sum_{k\in\mathbb{N}} \binom{0}{2k} q^k = \sum_{k\in\mathbb{N}} \begin{cases} 1, & \text{if } k=0; \\ 0, & \text{if } k\neq 0 \end{cases} \cdot q^k \qquad \left(\text{since } \binom{0}{2k} = \begin{cases} 1, & \text{if } 2k=0; \\ 0, & \text{if } 2k\neq 0 \end{cases} = \begin{cases} 1, & \text{if } k=0; \\ 0, & \text{if } k\neq 0 \end{cases}\right)$$
$$= 1\cdot q^0 = 1\cdot 1 = 1$$

[1] A sum of the form $\sum\limits_{k\in\mathbb{N}} a(k)$ (where $a : \mathbb{N} \to \mathbb{R}$ is some map) only makes sense if all but finitely many $k \in \mathbb{N}$ satisfy $a(k) = 0$. But this condition is easily verified for our sum $\sum\limits_{k\in\mathbb{N}} \binom{n}{2k} q^k$ (in fact, all $k \in \mathbb{N} \setminus \{0, 1, ..., n\}$ satisfy $k > n$, thus $2k \geq k > n$, thus $\binom{n}{2k} = 0$, thus $\binom{n}{2k} q^k = 0$; thus, all but finitely many $k \in \mathbb{N}$ satisfy $\binom{n}{2k} q^k = 0$) and (similarly) for our sum $\sum\limits_{k\in\mathbb{N}} \binom{n}{2k+1} q^k$. Similar arguments can show that all other sums of the form $\sum\limits_{k\in\mathbb{N}} a(k)$ that we will encounter in our solution will be well-defined.

and

$$g_0 = \sum_{k\in\mathbb{N}} \binom{0}{2k+1} q^k = \sum_{k\in\mathbb{N}} 0 \cdot q^k \qquad \left(\begin{array}{c}\text{since } \binom{0}{2k+1} = \begin{cases} 1, & \text{if } 2k+1=0; \\ 0, & \text{if } 2k+1\neq 0\end{cases} = 0, \\ \text{because } k\in\mathbb{N} \text{ yields } 2k+1\neq 0\end{array}\right)$$
$$= 0,$$

so that

$$\begin{pmatrix} f_0 & qg_0 \\ g_0 & f_0 \end{pmatrix} = \begin{pmatrix} 1 & q\cdot 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^0 .$$

In other words, (1) holds for $n = 0$. This completes the induction base.

Induction step. Let $N \in \mathbb{N}$. Assume that (1) holds for $n = N$. We have to show that (1) holds for $n = N + 1$ as well.

Since (1) holds for $n = N$, we have

$$\begin{pmatrix} f_N & qg_N \\ g_N & f_N \end{pmatrix} = \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^N .$$

But

$$f_{N+1} = \sum_{k\in\mathbb{N}} \binom{N+1}{2k} q^k = \sum_{k\in\mathbb{N}} \left(\binom{N}{2k} + \binom{N}{2k-1}\right) q^k$$
$$\left(\text{as } \binom{N+1}{2k} = \binom{N}{2k} + \binom{N}{2k-1} \text{ by the recurrence of the binomial coefficients}\right)$$
$$= \sum_{k\in\mathbb{N}} \binom{N}{2k} q^k + \sum_{k\in\mathbb{N}} \binom{N}{2k-1} q^k = \sum_{k\in\mathbb{N}} \binom{N}{2k} q^k + \sum_{\substack{k\in\mathbb{N};\\ k\geq 1}} \binom{N}{2k-1} q^k$$
$$\left(\begin{array}{c}\text{here we replaced the } \sum\limits_{k\in\mathbb{N}} \text{ sign by an } \sum\limits_{\substack{k\in\mathbb{N};\\ k\geq 1}} \text{ sign, since the addend for } k=0 \text{ is zero} \\ \left(\text{as } \binom{N}{2k-1} = \binom{N}{2\cdot 0-1} = \binom{N}{-1} = 0 \text{ for } k=0\right)\end{array}\right)$$
$$= \sum_{k\in\mathbb{N}} \binom{N}{2k} q^k + \sum_{k\in\mathbb{N}} \underbrace{\binom{N}{2(k+1)-1}}_{=\binom{N}{2k+1}} \underbrace{q^{k+1}}_{=q\cdot q^k}$$
$$\text{(here we substituted } k+1 \text{ for } k \text{ in the second sum)}$$
$$= \underbrace{\sum_{k\in\mathbb{N}} \binom{N}{2k} q^k}_{=f_N} + q \underbrace{\sum_{k\in\mathbb{N}} \binom{N}{2k+1} q^k}_{=g_N} = 1 \cdot f_N + q \cdot g_N$$

and

$$g_{N+1} = \sum_{k\in\mathbb{N}} \binom{N+1}{2k+1} q^k = \sum_{k\in\mathbb{N}} \left(\binom{N}{2k} + \binom{N}{2k+1} \right) q^k$$

$$\left(\text{as } \binom{N+1}{2k+1} = \binom{N}{2k} + \binom{N}{2k+1} \text{ by the recurrence of the binomial coefficients} \right)$$

$$= \underbrace{\sum_{k\in\mathbb{N}} \binom{N}{2k} q^k}_{=f_N} + \underbrace{\sum_{k\in\mathbb{N}} \binom{N}{2k+1} q^k}_{=g_N} = 1 \cdot f_N + 1 \cdot g_N,$$

so that

$$\begin{pmatrix} f_{N+1} & qg_{N+1} \\ g_{N+1} & f_{N+1} \end{pmatrix} = \begin{pmatrix} 1 \cdot f_N + q \cdot g_N & q\left(1 \cdot f_N + 1 \cdot g_N\right) \\ 1 \cdot f_N + 1 \cdot g_N & 1 \cdot f_N + q \cdot g_N \end{pmatrix} = \begin{pmatrix} 1 \cdot f_N + q \cdot g_N & 1 \cdot qg_N + q \cdot f_N \\ 1 \cdot f_N + 1 \cdot g_N & 1 \cdot qg_N + 1 \cdot f_N \end{pmatrix}$$

$$= \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix} \begin{pmatrix} f_N & qg_N \\ g_N & f_N \end{pmatrix} = \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^N = \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^{N+1}.$$

In other words, (1) holds for $n = N+1$. This completes the induction step. Thus, the induction proof is complete, so that (1) is proven for all $n \in \mathbb{N}$.

Now, (2) follows from

$$f_n^2 - qg_n^2 = f_n \cdot f_n - qg_n \cdot g_n = \det \begin{pmatrix} f_n & qg_n \\ g_n & f_n \end{pmatrix} = \det \left(\begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^n \right) \qquad \text{(by (1))}$$

$$= \left(\det \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix} \right)^n = (1 \cdot 1 - q \cdot 1)^n = (1-q)^n.$$

For any $a \in \mathbb{N}$ and $b \in \mathbb{N}$, we have

$$\begin{pmatrix} f_{a+b} & qg_{a+b} \\ g_{a+b} & f_{a+b} \end{pmatrix} = \begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^{a+b} \qquad \text{(by (1), applied to } n = a+b\text{)}$$

$$= \underbrace{\begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^a}_{\substack{=\begin{pmatrix} f_a & qg_a \\ g_a & f_a \end{pmatrix} \\ \text{(by (1), applied} \\ \text{to } n=a)}} \cdot \underbrace{\begin{pmatrix} 1 & q \\ 1 & 1 \end{pmatrix}^b}_{\substack{=\begin{pmatrix} f_b & qg_b \\ g_b & f_b \end{pmatrix} \\ \text{(by (1), applied} \\ \text{to } n=b)}}$$

$$= \begin{pmatrix} f_a & qg_a \\ g_a & f_a \end{pmatrix} \cdot \begin{pmatrix} f_b & qg_b \\ g_b & f_b \end{pmatrix} = \begin{pmatrix} f_a \cdot f_b + qg_a \cdot g_b & f_a \cdot qg_b + qg_a \cdot f_b \\ g_a \cdot f_b + f_a \cdot g_b & g_a \cdot qg_b + f_a \cdot f_b \end{pmatrix}$$

$$= \begin{pmatrix} f_a f_b + qg_a g_b & q\left(f_a g_b + g_a f_b\right) \\ f_a g_b + g_a f_b & f_a f_b + qg_a g_b \end{pmatrix}.$$

Thus, $f_{a+b} = f_a f_b + qg_a g_b$ and $g_{a+b} = f_a g_b + g_a f_b$, so that (4) and (5) are proven. For every $n \in \mathbb{N}$, we have

$$g_{2n} = g_{n+n} = f_n g_n + g_n f_n \qquad \text{(by (5))}$$

$$= 2f_n g_n,$$

and (3) follows.

Altogether, we have now proven Theorem 1.

From now on, we set $q = 2$. Then,

$$\begin{aligned}
f_n^2 g_n^2 &= (f_n g_n)^2 = \frac{1}{4}(2 f_n g_n)^2 = \frac{1}{4} g_{2n}^2 \qquad \text{(by (3))} \\
&= \frac{1}{4} \cdot \frac{1}{q} \cdot q g_{2n}^2 = \frac{1}{4} \cdot \frac{1}{q} \cdot \left(f_{2n}^2 - \left(f_{2n}^2 - q g_{2n}^2\right)\right) \\
&= \frac{1}{4} \cdot \frac{1}{q} \cdot \left(f_{2n}^2 - (1-q)^{2n}\right) \\
&\qquad \left(\text{since } f_{2n}^2 - q g_{2n}^2 = (1-q)^{2n}, \text{ what results if we substitute } 2n \text{ for } n \text{ in (2)}\right) \\
&= \frac{1}{4} \cdot \frac{1}{2} \cdot \left(f_{2n}^2 - \underbrace{(1-2)^{2n}}_{\substack{=(-1)^{2n}=1, \\ \text{since } 2n \text{ is even}}}\right) = \frac{1}{8}\left(f_{2n}^2 - 1\right) = \frac{1}{8}(f_{2n}-1)(f_{2n}+1) \\
&= \frac{1}{2} \cdot \frac{f_{2n}-1}{2} \cdot \frac{f_{2n}+1}{2} = \frac{1}{2} \cdot \frac{f_{2n}-1}{2} \cdot \left(\frac{f_{2n}-1}{2} + 1\right)
\end{aligned}$$

for every $n \in \mathbb{N}$. Since $\frac{f_{2n}-1}{2} \in \mathbb{Z}$ for every $n \in \mathbb{N}$ (since

$$\begin{aligned}
\frac{f_{2n}-1}{2} &= \frac{\sum\limits_{k \in \mathbb{N}} \binom{2n}{2k} q^k - 1}{2} = \frac{\sum\limits_{k \in \mathbb{N}} \binom{2n}{2k} 2^k - 1}{2} = \frac{\left(\binom{2n}{2 \cdot 0} 2^0 + \sum\limits_{\substack{k \in \mathbb{N}; \\ k \geq 1}} \binom{2n}{2k} 2^k\right) - 1}{2} \\
&= \frac{\left(1 + \sum\limits_{\substack{k \in \mathbb{N}; \\ k \geq 1}} \binom{2n}{2k} 2^k\right) - 1}{2} \qquad \left(\text{since } \binom{2n}{2 \cdot 0} 2^0 = \binom{2n}{0} 2^0 = 1 \cdot 1 = 1\right) \\
&= \frac{\sum\limits_{\substack{k \in \mathbb{N}; \\ k \geq 1}} \binom{2n}{2k} 2^k}{2} = \sum_{\substack{k \in \mathbb{N}; \\ k \geq 1}} \binom{2n}{2k} 2^{k-1} \in \mathbb{Z}
\end{aligned}$$

), this yields that $f_n^2 g_n^2$ is a triangular number for every $n \in \mathbb{N}$. This is exactly what the problem asked us to prove.

Remark. Theorem 1 could be proved more quickly using the binomial formula applied to $\left(1 + \sqrt{q}\right)^n$ and $\left(1 - \sqrt{q}\right)^n$. However, such a proof would fail if we replace $\mathbb{R}$ by a field of characteristic 2 and q is a square in that field. The proof given above works over any field and for any q. (Then again, from a deeper viewpoint, it is just a straightforward elementarization of the proof using the binomial formula.)