

Math Time problem proposal #1 ("The Dance of the Sigmas")

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Problem. Let $x_1, x_2, \dots, x_n$ be real numbers such that $x_1 + x_2 + \dots + x_n = 1$ and such that $x_i < 1$ for every $i \in \{1, 2, \dots, n\}$. Prove that

$$\sum_{1 \leq i < j \leq n} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} \geq \frac{n}{2(n-1)}.$$

Solution. First, for the sake of brevity¹, we introduce a notation: If $i_1, i_2, \dots, i_k$ are some free variables, if f is a function of k variables, and if $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_l$ are some assertions containing some of the variables $i_1, i_2, \dots, i_k$, then

$$\sum_{\substack{i_1, i_2, \dots, i_k \\ \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_l}} f(i_1, i_2, \dots, i_k)$$

will mean the sum of the values $f(i_1, i_2, \dots, i_k)$ over all "good" k -tuples $(i_1, i_2, \dots, i_k)$; hereby, a k -tuple $(i_1, i_2, \dots, i_k)$ is called "good" if it satisfies $i_u \in \{1, 2, \dots, n\}$ for every $u \in \{1, 2, \dots, k\}$ and also satisfies all the assertions $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_l$.

Using this notation, we can write many typical sums in a simpler form: For instance,

$$\sum_{1 \leq i \leq n} f(i) = \sum_{i=1}^n f(i), \text{ and } \sum_{1 \leq i < j \leq n} f(i, j) = \sum_{i < j}^{i, j} f(i, j).$$

Note that we won't use the notation $\sum_{k=u}^v f(k)$ for $f(u) + f(u+1) + \dots + f(v)$ in this solution, since it (theoretically) could be misunderstood because $\sum_{k=u}^v f(k)$ has a different meaning in our above notation.

With the notation defined above, we have

$$\sum_{1 \leq i < j \leq n} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} = \sum_{i < j}^{i, j} \frac{x_i x_j}{(1 - x_i)(1 - x_j)}.$$

Hence, the inequality in question,

$$\sum_{1 \leq i < j \leq n} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} \geq \frac{n}{2(n-1)},$$

rewrites as

$$\sum_{i < j}^{i, j} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} \geq \frac{n}{2(n-1)}.$$

Multiplication by $2(n-1)^2$ transforms this inequality into

$$(n-1)^2 \cdot 2 \sum_{i < j}^{i, j} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} \geq n(n-1).$$

¹In as far as brevity is possible for this solution...

Since

$$\begin{aligned}
2 \sum_{i < j} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} &= \sum_{i < j} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} + \sum_{i < j} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} \\
&= \sum_{i < j} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} + \sum_{j < i} \frac{x_j x_i}{(1 - x_j)(1 - x_i)} \\
&\quad \text{(at this step we have interchanged } i \text{ with } j \text{ in the second sum)} \\
&= \sum_{i < j} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} + \sum_{j < i} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} \\
&= \sum_{i \neq j} \frac{x_i x_j}{(1 - x_i)(1 - x_j)},
\end{aligned}$$

this inequality rewrites as

$$(n - 1)^2 \cdot \sum_{i \neq j} \frac{x_i x_j}{(1 - x_i)(1 - x_j)} \geq n(n - 1). \quad (1)$$

Thus, in order to solve the problem, it remains to prove this inequality (1).

Set $t_i = 1 - x_i$ for every $i \in \{1, 2, \dots, n\}$. Then, $t_i > 0$ for every $i \in \{1, 2, \dots, n\}$ (since $x_i < 1$ and thus $1 - x_i > 0$, so that $t_i = 1 - x_i > 0$). Besides, $t_i = 1 - x_i$ yields $x_i = 1 - t_i$. Finally,

$$\sum_{k=1}^n t_k = \sum_{k=1}^n (1 - x_k) = n - \sum_{k=1}^n x_k = n - (x_1 + x_2 + \dots + x_n) = n - 1. \quad (2)$$

Consequently,

$$\begin{aligned}
(n-1)^2 \cdot \sum_{i \neq j}^{i, j} \frac{x_i x_j}{(1-x_i)(1-x_j)} &= \sum_{i \neq j}^{i, j} \frac{(n-1)x_i \cdot (n-1)x_j}{(1-x_i)(1-x_j)} \\
&= \sum_{i \neq j}^{i, j} \frac{(n-1)(1-t_i) \cdot (n-1)(1-t_j)}{t_i t_j} = \sum_{i \neq j}^{i, j} \frac{((n-1) - (n-1)t_i) \cdot ((n-1) - (n-1)t_j)}{t_i t_j} \\
&= \sum_{i \neq j}^{i, j} \frac{\left(\sum_{k=1}^k t_k - (n-1)t_i \right) \cdot \left(\sum_{m=1}^m t_m - (n-1)t_j \right)}{t_i t_j} \\
&\quad \left(\text{since (2) yields } n-1 = \sum_{k=1}^k t_k \text{ and } n-1 = \sum_{m=1}^m t_m \right) \\
&= \sum_{i \neq j}^{i, j} \frac{\left(\left(t_j + \sum_{k \neq j}^k t_k \right) - (n-1)t_i \right) \cdot \left(\left(t_i + \sum_{m \neq i}^m t_m \right) - (n-1)t_j \right)}{t_i t_j} \\
&= \sum_{i \neq j}^{i, j} \frac{\left(\left(t_j + \sum_{k \neq j}^k t_k \right) - \sum_{k \neq j}^k t_i \right) \cdot \left(\left(t_i + \sum_{m \neq i}^m t_m \right) - \sum_{m \neq i}^m t_j \right)}{t_i t_j} \\
&= \sum_{i \neq j}^{i, j} \frac{\left(t_j + \sum_{k \neq j}^k (t_k - t_i) \right) \cdot \left(t_i + \sum_{m \neq i}^m (t_m - t_j) \right)}{t_i t_j} \\
&= \sum_{i \neq j}^{i, j} \frac{t_i t_j + \sum_{k \neq j}^k (t_k - t_i) \cdot t_i + \sum_{m \neq i}^m (t_m - t_j) \cdot t_j + \sum_{k \neq j}^k (t_k - t_i) \cdot \sum_{m \neq i}^m (t_m - t_j)}{t_i t_j} \\
&= \sum_{i \neq j}^{i, j} \left(1 + \frac{\sum_{k \neq j}^k (t_k - t_i)}{t_j} + \frac{\sum_{m \neq i}^m (t_m - t_j)}{t_i} + \frac{\sum_{k \neq j}^k (t_k - t_i) \cdot \sum_{m \neq i}^m (t_m - t_j)}{t_i t_j} \right) \\
&= \sum_{i \neq j}^{i, j} 1 + \sum_{i \neq j}^{i, j} \frac{\sum_{k \neq j}^k (t_k - t_i)}{t_j} + \sum_{i \neq j}^{i, j} \frac{\sum_{m \neq i}^m (t_m - t_j)}{t_i} + \sum_{i \neq j}^{i, j} \frac{\sum_{k \neq j}^k (t_k - t_i) \cdot \sum_{m \neq i}^m (t_m - t_j)}{t_i t_j}. \tag{3}
\end{aligned}$$

Now,

$$\sum_{i \neq j}^{i, j} 1 = n(n-1)$$

(because $n(n-1)$ is the number of all pairs (i, j) with i and j being elements of the

set $\{1, 2, \dots, n\}$ and satisfying $i \neq j$). Besides,

$$\begin{aligned}
& \sum_{i \neq j} \frac{\sum_{k \neq j}^k (t_k - t_i)}{t_j} + \sum_{i \neq j} \frac{\sum_{m \neq i}^m (t_m - t_j)}{t_i} = \sum_{i \neq j} \frac{\sum_{m \neq i}^m (t_m - t_j)}{t_i} + \sum_{i \neq j} \frac{\sum_{m \neq i}^m (t_m - t_j)}{t_i} \\
& \quad \text{(here we have interchanged } i \text{ with } j \text{ and renamed } k \text{ by } m \text{ in the first sum)} \\
& = 2 \sum_{i \neq j} \frac{\sum_{m \neq i}^m (t_m - t_j)}{t_i} = 2 \sum_i \frac{1}{t_i} \sum_{i \neq j}^j \sum_{m \neq i}^m (t_m - t_j) = 2 \sum_i \frac{1}{t_i} \left(\sum_{i \neq j}^j \sum_{m \neq i}^m t_m - \sum_{i \neq j}^j \sum_{m \neq i}^m t_j \right) \\
& = 2 \sum_i \frac{1}{t_i} \left(\sum_{m \neq i}^m \sum_{i \neq j}^j t_m - \sum_{i \neq j}^j \sum_{m \neq i}^m t_j \right) = 2 \sum_i \frac{1}{t_i} \left(\sum_{m \neq i}^m (n-1) t_m - \sum_{i \neq j}^j (n-1) t_j \right) \\
& = 2 \sum_i \frac{1}{t_i} \left(\sum_{j \neq i}^j (n-1) t_j - \sum_{i \neq j}^j (n-1) t_j \right) \quad \text{(here we have renamed } m \text{ by } j \text{ in the second sum)} \\
& = 2 \sum_i \frac{1}{t_i} \left(\sum_{i \neq j}^j (n-1) t_j - \sum_{i \neq j}^j (n-1) t_j \right) = 2 \sum_i \frac{1}{t_i} \cdot 0 = 0.
\end{aligned}$$

Hence, (3) becomes

$$\begin{aligned}
& (n-1)^2 \cdot \sum_{i \neq j} \frac{x_i x_j}{(1-x_i)(1-x_j)} \\
& = \underbrace{\sum_{i \neq j} 1}_{=n(n-1)} + \underbrace{\sum_{i \neq j} \frac{\sum_{k \neq j}^k (t_k - t_i)}{t_j} + \sum_{i \neq j} \frac{\sum_{m \neq i}^m (t_m - t_j)}{t_i}}_{=0} + \sum_{i \neq j} \frac{\sum_{k \neq j}^k (t_k - t_i) \cdot \sum_{m \neq i}^m (t_m - t_j)}{t_i t_j} \\
& = n(n-1) + \sum_{i \neq j} \frac{\sum_{k \neq j}^k (t_k - t_i) \cdot \sum_{m \neq i}^m (t_m - t_j)}{t_i t_j}.
\end{aligned}$$

Therefore, the inequality that we have to prove, namely (1), rewrites as

$$n(n-1) + \sum_{i \neq j} \frac{\sum_{k \neq j}^k (t_k - t_i) \cdot \sum_{m \neq i}^m (t_m - t_j)}{t_i t_j} \geq n(n-1).$$

This is obviously equivalent to

$$\sum_{i \neq j} \frac{\sum_{k \neq j}^k (t_k - t_i) \cdot \sum_{m \neq i}^m (t_m - t_j)}{t_i t_j} \geq 0. \quad (4)$$

Thus, we only have to prove (4) in order to complete the solution of the problem (because (4) is equivalent to (1), and proving (1) solves the problem).

Now,

$$\begin{aligned}
& \sum_{i \neq j} \frac{\sum_{k \neq j}^k (t_k - t_i) \cdot \sum_{m \neq i}^m (t_m - t_j)}{t_i t_j} = \sum_{i \neq j, k \neq j, m \neq i}^{i, j, k, m} \frac{(t_k - t_i) (t_m - t_j)}{t_i t_j} \\
&= \sum_{i \neq j, k \neq j, m \neq i, k \neq i, m \neq j}^{i, j, k, m} \frac{(t_k - t_i) (t_m - t_j)}{t_i t_j} + \sum_{i \neq j, k \neq j, m \neq i, (k=i \vee m=j)}^{i, j, k, m} \underbrace{\frac{(t_k - t_i) (t_m - t_j)}{t_i t_j}}_{=0, \text{ since we have } k=i \text{ or } m=j \text{ and thus } t_k - t_i = 0 \text{ or } t_m - t_j = 0} \\
&\quad \left(\begin{array}{c} \text{because every quadruple } (i, j, k, m) \text{ satisfies exactly one of the two assertions} \\ (k \neq i \wedge m \neq j) \text{ and } (k = i \vee m = j) \end{array} \right) \\
&= \sum_{i \neq j, k \neq j, m \neq i, k \neq i, m \neq j}^{i, j, k, m} \frac{(t_k - t_i) (t_m - t_j)}{t_i t_j} + 0 = \sum_{i \neq j, k \neq j, m \neq i, k \neq i, m \neq j}^{i, j, k, m} \frac{(t_k - t_i) (t_m - t_j)}{t_i t_j} \\
&= \sum_{i \neq j, k \neq j, m \neq i, k \neq i, m \neq j, k \neq m}^{i, j, k, m} \frac{(t_k - t_i) (t_m - t_j)}{t_i t_j} + \sum_{i \neq j, k \neq j, m \neq i, k \neq i, m \neq j, k=m}^{i, j, k, m} \underbrace{\frac{(t_k - t_i) (t_m - t_j)}{t_i t_j}}_{= \frac{(t_k - t_i) (t_k - t_j)}{t_i t_j} \text{ since } k=m} \\
&= \sum_{i \neq j, k \neq j, m \neq i, k \neq i, m \neq j, k \neq m}^{i, j, k, m} \frac{(t_k - t_i) (t_m - t_j)}{t_i t_j} + \sum_{i \neq j, k \neq j, k \neq i}^{i, j, k} \frac{(t_k - t_i) (t_k - t_j)}{t_i t_j}.
\end{aligned}$$

Thus, in order to prove (4), it is enough to prove the two inequalities

$$\sum_{i \neq j, k \neq j, m \neq i, k \neq i, m \neq j, k \neq m}^{i, j, k, m} \frac{(t_k - t_i) (t_m - t_j)}{t_i t_j} \geq 0; \quad (5)$$

$$\sum_{i \neq j, k \neq j, k \neq i}^{i, j, k} \frac{(t_k - t_i) (t_k - t_j)}{t_i t_j} \geq 0. \quad (6)$$

Proving (5) is rather easy: First, the six conditions $i \neq j$, $k \neq j$, $m \neq i$, $k \neq i$, $m \neq j$, $k \neq m$ altogether (connected by a logical "and") are equivalent to the condition that the four numbers i, j, k, m are pairwise distinct, i. e. that the set $\{i, j, k, m\}$

has exactly 4 elements, i. e. that $|\{i, j, k, m\}| = 4$. Hence,

$$\begin{aligned}
& \sum_{i \neq j, k \neq j, m \neq i, k \neq i, m \neq j, k \neq m}^{i, j, k, m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} = \sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} \\
&= \frac{1}{4} \cdot \left(\sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} + \sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} \right. \\
&\quad \left. + \sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} + \sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} \right) \\
&= \frac{1}{4} \cdot \left(\sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} + \sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} \frac{(t_i - t_k)(t_m - t_j)}{t_k t_j} \right. \\
&\quad \left. + \sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} \frac{(t_k - t_i)(t_j - t_m)}{t_i t_m} + \sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} \frac{(t_i - t_k)(t_j - t_m)}{t_k t_m} \right) \\
&\quad \left(\begin{array}{l} \text{at this step, we have renamed some variables: in the second sum, we} \\ \text{interchanged } i \text{ with } k; \text{ in the third sum, we interchanged } j \text{ with } m; \text{ in the} \\ \text{fourth sum, we interchanged both } i \text{ with } k \text{ and } j \text{ with } m \end{array} \right) \\
&= \frac{1}{4} \cdot \sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} \left(\frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} + \frac{(t_i - t_k)(t_m - t_j)}{t_k t_j} + \frac{(t_k - t_i)(t_j - t_m)}{t_i t_m} + \frac{(t_i - t_k)(t_j - t_m)}{t_k t_m} \right) \\
&= \frac{1}{4} \cdot \sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} \left(\frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} - \frac{(t_k - t_i)(t_m - t_j)}{t_k t_j} - \frac{(t_k - t_i)(t_m - t_j)}{t_i t_m} + \frac{(t_k - t_i)(t_m - t_j)}{t_k t_m} \right) \\
&= \frac{1}{4} \cdot \sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} (t_k - t_i)(t_m - t_j) \cdot \left(\frac{1}{t_i t_j} - \frac{1}{t_k t_j} - \frac{1}{t_i t_m} + \frac{1}{t_k t_m} \right) \\
&= \frac{1}{4} \cdot \sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} (t_k - t_i)(t_m - t_j) \cdot \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j t_k t_m} = \frac{1}{4} \cdot \sum_{|\{i, j, k, m\}|=4}^{i, j, k, m} \frac{(t_k - t_i)^2 (t_m - t_j)^2}{t_i t_j t_k t_m} \geq 0
\end{aligned}$$

because squares are nonnegative; thus, (5) is proven.

In order to prove (6), we proceed similarly: First, the three conditions $i \neq j$, $k \neq j$, $k \neq i$ altogether (connected by a logical "and") are equivalent to the condition that the three numbers i, j, k are pairwise distinct, i. e. that the set $\{i, j, k\}$ has exactly

3 elements, i. e. that $|\{i, j, k\}| = 3$. Hence,

$$\begin{aligned}
& \sum_{i \neq j, k \neq j, k \neq i}^{i, j, k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} = \sum_{|\{i, j, k\}|=3}^{i, j, k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} \\
&= \frac{1}{3} \cdot \left(\sum_{|\{i, j, k\}|=3}^{i, j, k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} + \sum_{|\{i, j, k\}|=3}^{i, j, k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} + \sum_{|\{i, j, k\}|=3}^{i, j, k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} \right) \\
&= \frac{1}{3} \cdot \left(\sum_{|\{i, j, k\}|=3}^{i, j, k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} + \sum_{|\{i, j, k\}|=3}^{i, j, k} \frac{(t_i - t_j)(t_i - t_k)}{t_j t_k} + \sum_{|\{i, j, k\}|=3}^{i, j, k} \frac{(t_j - t_k)(t_j - t_i)}{t_k t_i} \right) \\
&\quad \left(\begin{array}{l} \text{at this step, we have renamed some variables:} \\ \text{in the second sum, we renamed } i, j, k \text{ by } j, k, i, \text{ respectively;} \\ \text{in the third sum, we renamed } i, j, k \text{ by } k, i, j, \text{ respectively} \end{array} \right) \\
&= \frac{1}{3} \cdot \sum_{|\{i, j, k\}|=3}^{i, j, k} \left(\frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} + \frac{(t_i - t_j)(t_i - t_k)}{t_j t_k} + \frac{(t_j - t_k)(t_j - t_i)}{t_k t_i} \right) \\
&= \frac{1}{3} \cdot \sum_{|\{i, j, k\}|=3}^{i, j, k} \frac{t_k(t_k - t_i)(t_k - t_j) + t_i(t_i - t_j)(t_i - t_k) + t_j(t_j - t_k)(t_j - t_i)}{t_i t_j t_k} \geq 0,
\end{aligned}$$

because the Schur inequality yields $t_k(t_k - t_i)(t_k - t_j) + t_i(t_i - t_j)(t_i - t_k) + t_j(t_j - t_k)(t_j - t_i) \geq 0$ for any triple (i, j, k) . Thus, the inequality (6) is proven.

As both inequalities (5) and (6) are verified now, the inequality (4) follows, and thus the problem is solved.

Remark. The above problem is a generalization of the Schur inequality (which states that $a(a - b)(a - c) + b(b - c)(b - a) + c(c - a)(c - b) \geq 0$ for any three non-negative reals a, b, c) to n variables (in fact, if you set $n = 3$ in the problem, you get the Schur inequality for the numbers $1 - x_1, 1 - x_2, 1 - x_3$).

In the particular case when the reals $x_1, x_2, \dots, x_n$ are nonnegative, the problem can be solved much more easily - a solution for this case was given in [1] (proof of Theorem 4.1).

References

- [1] Darij Grinberg, *An inequality involving $2n$ numbers*.