An algebraic approach to Hall's matching theorem – abridged version

Darij Grinberg

6 October 2007 (un-proofread)

The purpose of this note is to present a proof of Hall's matching theorem (also called marriage theorem) which I have not encountered elsewhere in literature - what yet does not mean that it is necessarily new.

We refer to Hall's theorem in the following form:

> **Theorem 1 (Hall).** Let n be a positive integer. Let Γ be a bipartite graph whose set of vertices consists of n blue vertices B_1, B_2, ..., B_n and n green vertices G_1, G_2, ..., G_n. Then, the graph Γ has a perfect matching if and only if every subset $J \subseteq \{1, 2, ..., n\}$ satisfies $\left| \bigcup\limits_{i \in J} \mathcal{N}(G_i) \right| \geq |J|$.

Some notations used in this theorem require *explanations*:

- A *bipartite* graph is a (simple, non-directed) graph with each vertex colored either green or blue such that every edge of the graph connects a blue vertex and a green vertex.
- A *perfect matching* of the bipartite graph Γ means a permutation ϕ of the set $\{1, 2, ..., n\}$ such that for every $j \in \{1, 2, ..., n\}$, the vertex B_j is connected to the vertex $G_{\phi(j)}$.
- The number of elements of a finite set X is denoted by $|X|$.
- Finally, if A is a vertex of our graph Γ, then a *neighbour* of A means any other vertex of Γ which is connected to A by an edge. We denote by $\mathcal{N}(A)$ the set of all neighbours of A.

Proofs of Theorem 1 abound in literature - see, e. g., Chapter 11 of [1], Theorem 12.2 in [2], or Theorem 2.1.2 in [3]. Here we are going to sketch a proof (a complete and detailed, yet horrible to read presentation of this proof can be found in [4]) which is longer than most of these, but applies an idea apparently new, and potentially interesting for further study.

Proof of Theorem 1. In order to show Theorem 1, we have to verify two assertions:

Assertion 1. If the graph Γ has a perfect matching, then every subset $J \subseteq \{1, 2, ..., n\}$ satisfies $\left| \bigcup\limits_{i \in J} \mathcal{N}(G_i) \right| \geq |J|$.

Assertion 2. If the graph Γ has no perfect matching, then there exists a subset $J \subseteq \{1, 2, ..., n\}$ which does *not* satisfy $\left|\bigcup_{i \in J} \mathcal{N}(G_i)\right| \geq |J|$.

The almost trivial *proof of Assertion 1* is left to the reader. The interesting part is the *proof of Assertion 2.* Before we come to this proof, we define some notations concerning matrices:

- For a matrix A, we denote by $A\left[\begin{array}{c} j \\ i \end{array}\right]$ the entry in the j-th column and the i-th row of A. [This is usually denoted by A_{ij}.]

- Let A be a matrix with u rows and v columns. Let $j_1, j_2, ..., j_k$ be some pairwisely distinct integers from the set $\{1, 2, ..., v\}$, and let $i_1, i_2, ..., i_l$ be some pairwisely distinct integers from the set $\{1, 2, ..., u\}$. Then, we denote by $A\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_l}\right]$ the matrix with l rows and k columns which is defined as follows: For any integers $p \in \{1, 2, ..., l\}$ and $q \in \{1, 2, ..., k\}$, we have $\left(A\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_l}\right]\right)\left[\begin{array}{c} q \\ p \end{array}\right] = A\left[\begin{array}{c} j_q \\ i_p \end{array}\right]$.

  Informally speaking, $A\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_l}\right]$ is the matrix formed by the intersections of the columns numbered $j_1, j_2, ..., j_k$ with the rows numbered $i_1, i_2, ..., i_l$ of the matrix A, but the order of these columns and rows depends on the order of the integers $j_1, j_2, ..., j_k$ and the order of the integers $i_1, i_2, ..., i_l$.

  Such a matrix $A\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_l}\right]$ is called a *minor* of the matrix A.

  Examples:

$$\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \end{pmatrix}\left[\frac{2,4}{1,3}\right] = \begin{pmatrix} b & d \\ b'' & d'' \end{pmatrix};$$
$$\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix}\left[\frac{3,1}{1,2,3}\right] = \begin{pmatrix} c & a \\ c' & a' \\ c'' & a'' \end{pmatrix}.$$

  Note that, thus, for any matrix A, the matrix $A\left[\frac{j}{i}\right]$ is the 1×1 matrix consisting of the only element $A\left[\begin{array}{c} j \\ i \end{array}\right]$.

- If m is a positive integer, and $r \in \{1, 2, ..., m\}$, then the notation $j_1, j_2, ..., \widehat{j_r}, ..., j_m$ is going to mean "the numbers $j_1, j_2, ..., j_m$ with the number j_r left out" (i. e. "the numbers $j_1, j_2, ..., j_{r-2}, j_{r-1}, j_{r+1}, j_{r+2}, ..., j_m$ ").

We will make use of a method of computing determinants called *developing a determinant along a row.* This method states that for any $k \times k$ matrix U and any

$s \in \{1, 2, ..., k\}$, we have

$$\det U = \sum_{r=1}^{k} (-1)^{s+r} \cdot U \begin{bmatrix} r \\ s \end{bmatrix} \cdot \det \left(U \begin{bmatrix} 1, 2, ..., \widehat{r}, ..., k \\ 1, 2, ..., \widehat{s}, ..., k \end{bmatrix} \right). \qquad (1)$$

Now, to our proof of Assertion 2. We assume that the graph Γ has no perfect matching. In order to prove Assertion 2, we have to find a subset $J \subseteq \{1, 2, ..., n\}$ which does *not* satisfy $\left| \bigcup_{i \in J} \mathcal{N}(G_i) \right| \geq |J|$.

Let K be an arbitrary field (for instance, $\mathbb{Q}$). Let L be the field of all rational functions of n^2 indeterminates $X_{1,1}$, $X_{1,2}$, ..., $X_{n,n}$ (one indeterminate $X_{i,j}$ for each pair $(i, j) \in \{1, 2, ..., n\}^2$) over K.

Then, $L = K(X_{1,1}, X_{1,2}, ..., X_{n,n})$.

We define a matrix $S \in \mathrm{M}_n(L)$ by setting

$$S \begin{bmatrix} j \\ i \end{bmatrix} = \begin{cases} X_{i,j}, & \text{if } G_j \in \mathcal{N}(B_i); \\ 0, & \text{if } G_j \notin \mathcal{N}(B_i) \end{cases} \qquad \text{for any two } i \text{ and } j \text{ from the set } \{1, 2, ..., n\}.$$

This matrix S stores all information about the bipartite graph Γ in it: For any blue vertex B_i and any green vertex G_j, we can tell whether B_i and G_j are connected from the entry $S \begin{bmatrix} j \\ i \end{bmatrix}$ of this matrix (in fact, the vertices B_i and G_j are connected if and only if $S \begin{bmatrix} j \\ i \end{bmatrix} \neq 0$).

Since the graph Γ has no perfect matching, it is easy to see that $\det S = 0$ (in fact, by the definition of the determinant as a sum over permutations, we have $\det S = \sum_{\pi \in S_n} \operatorname{sign} \pi \cdot \prod_{i=1}^{n} S \begin{bmatrix} \pi(i) \\ i \end{bmatrix}$, but for any permutation $\pi \in S_n$ the product $\prod_{i=1}^{n} S \begin{bmatrix} \pi(i) \\ i \end{bmatrix}$ has at least one of its factors being equal to 0, because otherwise this permutation π would be a perfect matching of Γ). Thus, the columns of the matrix S are linearly dependent.

Therefore, we can find a minimal family of linearly dependent columns of the matrix S. That means, we can find a subset $\{j_1, j_2, ..., j_k\}$ of $\{1, 2, ..., n\}$ such that the k columns of the matrix S numbered j_1, j_2, ..., j_k are linearly dependent, but for every $u \in \{1, 2, ..., k\}$, the $k-1$ columns of the matrix S numbered j_1, j_2, ..., $\widehat{j_u}$, ..., j_k are linearly independent.

It is easy to conclude from this that the matrix $S \begin{bmatrix} j_1, j_2, ..., j_k \\ 1, 2, ..., n \end{bmatrix}$ has rank $k-1$. Hence, this matrix has $k-1$ linearly independent rows, and every row of this matrix is a linear combination of these $k-1$ rows.

So let the rows numbered i_1, i_2, ..., i_{k-1} be $k-1$ linearly independent rows of the matrix $S \begin{bmatrix} j_1, j_2, ..., j_k \\ 1, 2, ..., n \end{bmatrix}$. Then, every row of the matrix $S \begin{bmatrix} j_1, j_2, ..., j_k \\ 1, 2, ..., n \end{bmatrix}$ is a linear combination of these $k-1$ rows i_1, i_2, ..., i_{k-1}. In other words, for every $i \in \{1, 2, ..., n\}$, there exist elements $\alpha_{i,\ 1}$, $\alpha_{i,\ 2}$, ..., $\alpha_{i,\ k-1}$ of L such that the i-th row of the matrix $S \begin{bmatrix} j_1, j_2, ..., j_k \\ 1, 2, ..., n \end{bmatrix}$ is the sum of $\alpha_{i,\ v}$ times the i_v-th row of this matrix over all $v \in$

$\{1,2,...,k-1\}$. This means that

$$S\left[\begin{array}{c} j_u \\ i \end{array}\right]=\sum_{v=1}^{k-1}\alpha_{i,\ v}S\left[\begin{array}{c} j_u \\ i_v \end{array}\right]$$

for every $u\in\{1,2,...,k\}$.

Now, we will show that for each $r\in\{1,2,...,k\}$, we have $\det\left(S\left[\dfrac{j_1,j_2,...,\widehat{j_r},...,j_k}{i_1,i_2,...,i_{k-1}}\right]\right)\neq 0$. In fact, assume that this is not the case. Then, there exists some $r\in\{1,2,...,k\}$ such that $\det\left(S\left[\dfrac{j_1,j_2,...,\widehat{j_r},...,j_k}{i_1,i_2,...,i_{k-1}}\right]\right)=0$. Hence, for this r, the $k-1$ columns of the matrix $S\left[\dfrac{j_1,j_2,...,\widehat{j_r},...,j_k}{i_1,i_2,...,i_{k-1}}\right]$ are linearly dependent. In other words, the columns j_1, j_2, ..., $\widehat{j_r}$, ..., j_k of the matrix $S\left[\dfrac{1,2,...,n}{i_1,i_2,...,i_{k-1}}\right]$ are linearly dependent. This means that there exist elements β_1, β_2, ..., $\widehat{\beta_r}$, ..., β_k of L which are not all equal to 0 such that the sum of β_u times the j_u-th column of the matrix $S\left[\dfrac{1,2,...,n}{i_1,i_2,...,i_{k-1}}\right]$ over all $u\in\{1,2,...,\widehat{r},...,k\}$ equals 0. Equivalently,

$$\sum_{1\leq u\leq k;\ u\neq r}\beta_u\cdot S\left[\begin{array}{c} j_u \\ i_v \end{array}\right]=0$$

for each $v\in\{1,2,...,k-1\}$. Then, for every $i\in\{1,2,...,n\}$, using the relation $S\left[\begin{array}{c} j_u \\ i \end{array}\right]=\sum\limits_{v=1}^{k-1}\alpha_{i,\ v}S\left[\begin{array}{c} j_u \\ i_v \end{array}\right]$ which holds for every $u\in\{1,2,...,k\}$, we obtain

$$\begin{aligned}\sum_{1\leq u\leq k;\ u\neq r}\beta_u\cdot S\left[\begin{array}{c} j_u \\ i \end{array}\right] &= \sum_{1\leq u\leq k;\ u\neq r}\beta_u\cdot\sum_{v=1}^{k-1}\alpha_{i,\ v}S\left[\begin{array}{c} j_u \\ i_v \end{array}\right] \\ &= \sum_{v=1}^{k-1}\alpha_{i,\ v}\underbrace{\sum_{1\leq u\leq k;\ u\neq r}\beta_u\cdot S\left[\begin{array}{c} j_u \\ i_v \end{array}\right]}_{=0}=0.\end{aligned}$$

In other words, the sum of β_u times the j_u-th column of the matrix S over all $u\in\{1,2,...,\widehat{r},...,k\}$ equals 0. This yields that the columns of the matrix S numbered j_1, j_2, ..., $\widehat{j_r}$, ..., j_k are linearly dependent. But this contradicts to the fact that for every $u\in\{1,2,...,k\}$, the $k-1$ columns of the matrix S numbered j_1, j_2, ..., $\widehat{j_u}$, ..., j_k are linearly independent. This contradiction yields that our assumption was wrong. Thus, we have proven that for each $r\in\{1,2,...,k\}$, we have

$$\det\left(S\left[\frac{j_1,j_2,...,\widehat{j_r},...,j_k}{i_1,i_2,...,i_{k-1}}\right]\right)\neq 0. \qquad (2)$$

Now let $J=\{j_1,j_2,...,j_k\}$. Then, we are going to prove that $\bigcup\limits_{i\in J}\mathcal{N}(G_i)\subseteq\left\{B_{i_1},B_{i_2},...,B_{i_{k-1}}\right\}$.

In fact, we are going to prove this by contradiction: Assume that $\bigcup_{i\in J}\mathcal{N}(G_i) \subseteq \{B_{i_1}, B_{i_2}, ..., B_{i_{k-1}}\}$ does not hold. Then, there exists a vertex T of the graph Γ which lies in $\bigcup_{i\in J}\mathcal{N}(G_i)$ but not in $\{B_{i_1}, B_{i_2}, ..., B_{i_{k-1}}\}$.

From $T \in \bigcup_{i\in J}\mathcal{N}(G_i)$, it follows that there exists some $j \in J$ with $T \in \mathcal{N}(G_j)$. Thus, T is a blue vertex of the graph Γ, so that $T = B_{\widetilde{i}}$ for some $\widetilde{i} \in \{1, 2, ..., n\}$. But since $T \notin \{B_{i_1}, B_{i_2}, ..., B_{i_{k-1}}\}$, we must have $\widetilde{i} \notin \{i_1, i_2, ..., i_{k-1}\}$.

Besides, since $j \in J = \{j_1, j_2, ..., j_k\}$, there exists an $q \in \{1, 2, ..., k\}$ such that $j = j_q$. Since $T = B_{\widetilde{i}}$ and $j = j_q$, the relation $T \in \mathcal{N}(G_j)$ becomes $B_{\widetilde{i}} \in \mathcal{N}(G_{j_q})$. Thus, $G_{j_q} \in \mathcal{N}(B_{\widetilde{i}})$. Hence,

$$S\begin{bmatrix} j_q \\ \widetilde{i} \end{bmatrix} = \begin{cases} X_{\widetilde{i},j_q}, & \text{if } G_{j_q} \in \mathcal{N}(B_{\widetilde{i}}); \\ 0, & \text{if } G_{j_q} \notin \mathcal{N}(B_{\widetilde{i}}) \end{cases} = X_{\widetilde{i},j_q}.$$

Since the numbers i_1, i_2, ..., i_{k-1} are pairwisely distinct (because the rows of the matrix $S\left[\frac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$ numbered i_1, i_2, ..., i_{k-1} are linearly independent) and we have $\widetilde{i} \notin \{i_1, i_2, ..., i_{k-1}\}$, we can conclude that the numbers i_1, i_2, ..., i_{k-1}, $\widetilde{i}$ are pairwisely distinct. Now consider the square matrix $S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]$. This matrix is a $k \times k$ matrix, but its rank is $\leq k-1$ (in fact, this matrix is a minor of the matrix $S\left[\frac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$, so its rank must be $\leq$ to the rank of $S\left[\frac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$, which is known to be $k-1$). Hence, the determinant of this matrix must be 0; that is,

$$\det\left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right) = 0. \tag{3}$$

But on the other hand, by developing the determinant of the matrix $S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]$ along its last (that is, its k-th) row (i. e., by applying the formula (1) to $U = S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]$ and $s = k$), we obtain

$$\begin{aligned} &\det\left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right) \\ = &\sum_{r=1}^{k}(-1)^{k+r}\cdot\left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right)\begin{bmatrix} r \\ k \end{bmatrix}\cdot\det\left(\left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right)\left[\frac{1, 2, ..., \widehat{r}, ..., k}{1, 2, ..., \widehat{k}, ..., k}\right]\right) \end{aligned}$$

(hereby, of course, 1, 2, ..., $\widehat{k}$, ..., k is just a complicated notation for 1, 2, ..., $k-1$). This monstrous equation simplifies to

$$\begin{aligned} &\det\left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right) \\ = &\sum_{r=1}^{k}(-1)^{k+r}\cdot S\begin{bmatrix} j_r \\ \widetilde{i} \end{bmatrix}\cdot\det\left(S\left[\frac{j_1, j_2, ..., \widehat{j_r}, ..., j_k}{i_1, i_2, ..., i_{k-1}}\right]\right). \end{aligned} \tag{4}$$

Denote $d_r = \det\left(S\left[\frac{j_1, j_2, ..., \widehat{j_r}, ..., j_k}{i_1, i_2, ..., i_{k-1}}\right]\right)$ for every $r \in \{1, 2, ..., k\}$. Then, (2) yields $d_r \neq 0$ for every $r \in \{1, 2, ..., k\}$, while (4) transforms into

$$\det\left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right) = \sum_{r=1}^{k} (-1)^{k+r} \cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right] \cdot d_r.$$

Comparing this with (3), we obtain

$$0 = \sum_{r=1}^{k} (-1)^{k+r} \cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right] \cdot d_r.$$

This rewrites as

$$0 = \sum_{1 \leq r \leq k;\ r \neq q} (-1)^{k+r} \cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right] \cdot d_r + (-1)^{k+q} \cdot S\left[\begin{array}{c} j_q \\ \widetilde{i} \end{array}\right] \cdot d_q.$$

Hence,

$$(-1)^{k+q} \cdot S\left[\begin{array}{c} j_q \\ \widetilde{i} \end{array}\right] \cdot d_q = -\sum_{1 \leq r \leq k;\ r \neq q} (-1)^{k+r} \cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right] \cdot d_r.$$

Since $(-1)^{k+q} \neq 0$ and $d_q \neq 0$ (because $d_r \neq 0$ for every $r \in \{1, 2, ..., k\}$), we can divide this equation by $(-1)^{k+q} \cdot d_q$, and obtain

$$S\left[\begin{array}{c} j_q \\ \widetilde{i} \end{array}\right] = \frac{-\sum\limits_{1 \leq r \leq k;\ r \neq q} (-1)^{k+r} \cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right] \cdot d_r}{(-1)^{k+q} \cdot d_q}. \tag{5}$$

Now, there are five easy facts:

$$\text{For every } r \in \{1, 2, ..., k\}, \text{ we have } (-1)^{k+r} \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i}, j_q}}, ..., X_{n,n}\right). \tag{6}$$

$$\text{We have } (-1)^{k+q} \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i}, j_q}}, ..., X_{n,n}\right). \tag{7}$$

$$\text{For every } r \in \{1, 2, ..., k\}, \text{ we have } d_r \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i}, j_q}}, ..., X_{n,n}\right). \tag{8}$$

$$\text{We have } d_q \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i}, j_q}}, ..., X_{n,n}\right). \tag{9}$$

$$\begin{array}{c} \text{For every } r \in \{1, 2, ..., k\} \text{ with } r \neq q, \text{ we have} \\ S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right] \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i}, j_q}}, ..., X_{n,n}\right). \end{array} \tag{10}$$

The proofs of these five facts are very easy: Firstly, (6) and (7) are trivial.

For the proof of (8), note that $d_r = \det\left(S\left[\frac{j_1, j_2, ..., \widehat{j_r}, ..., j_k}{i_1, i_2, ..., i_{k-1}}\right]\right)$ is the determinant of the matrix $S\left[\frac{j_1, j_2, ..., \widehat{j_r}, ..., j_k}{i_1, i_2, ..., i_{k-1}}\right]$ whose entries all have the form $S\left[\begin{array}{c} j_x \\ i_y \end{array}\right] =$

$\left\{ \begin{array}{c} X_{i_y,j_x}, \text{ if } G_{j_x} \in \mathcal{N}\left(B_{i_y}\right); \\ 0, \text{ if } G_{j_x} \notin \mathcal{N}\left(B_{i_y}\right) \end{array} \right.$ and thus lie in $K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right)$ (because X_{i_y,j_x} lies in $K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right)$, since $\widetilde{i} \notin \{i_1, i_2, ..., i_{k-1}\}$ yields $i_y \neq \widetilde{i}$). Hence, the determinant d_r of this matrix also lies in $K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right)$, and (8) is proven.

The relation (9) obviously follows from (8).

The relation (10) follows from $S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right] = \left\{ \begin{array}{c} X_{\widetilde{i},j_r}, \text{ if } G_{j_r} \in \mathcal{N}\left(B_{\widetilde{i}}\right); \\ 0, \text{ if } G_{j_r} \notin \mathcal{N}\left(B_{\widetilde{i}}\right) \end{array} \right.$ and $X_{\widetilde{i},j_r} \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right)$ (the latter because $r \neq q$ yields $j_r \neq j_q$).

From (6), (7), (8), (9) and (10) together, it follows that

$$\frac{-\sum\limits_{1 \leq r \leq k;\ r \neq q} (-1)^{k+r} \cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right] \cdot d_r}{(-1)^{k+q} \cdot d_q} \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right).$$

Using (5), this transforms into $S\left[\begin{array}{c} j_q \\ \widetilde{i} \end{array}\right] \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right)$. But this is wrong, because we know that $S\left[\begin{array}{c} j_q \\ \widetilde{i} \end{array}\right] = X_{\widetilde{i},j_q} \notin K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right)$. Hence, we have obtained a contradiction.

This contradiction shows that our assumption was wrong. Hence, we do have $\bigcup\limits_{i \in J} \mathcal{N}(G_i) \subseteq \left\{B_{i_1}, B_{i_2}, ..., B_{i_{k-1}}\right\}$. Thus, $\left|\bigcup\limits_{i \in J} \mathcal{N}(G_i)\right| \leq \left|\left\{B_{i_1}, B_{i_2}, ..., B_{i_{k-1}}\right\}\right| = k-1$. But $|J| = |\{j_1, j_2, ..., j_k\}| = k$. Hence, $\left|\bigcup\limits_{i \in J} \mathcal{N}(G_i)\right| \leq k-1 < k = |J|$.

Thus, the subset $J \subseteq \{1, 2, ..., n\}$ does *not* satisfy $\left|\bigcup\limits_{i \in J} \mathcal{N}(G_i)\right| \geq |J|$. This proves Assertion 2, and therefore completes the proof of Theorem 1.

References

[1] George Pólya, Robert E. Tarjan, Donald R. Woods, *Notes on Introductory Combinatorics*, Boston/Basel/Stuttgart 1983.

[2] L. Lovász, K. Vesztergombi, *Discrete Mathematics*, lecture notes, 1999.
`http://www.cs.tau.ac.il/~odedr/teaching/discrete_math_fall_2005/dmbook.pdf`

[3] Reinhard Diestel, *Graph Theory*, 3rd Edition, Heidelberg 2005.
`http://www.math.uni-hamburg.de/home/diestel/books/graph.theory/`

[4] Darij Grinberg, *An algebraic approach to Hall's matching theorem*.
`http://www.stud.uni-muenchen.de/~darij.grinberg/`