An algebraic approach to Hall's matching theorem

Darij Grinberg

6 October 2007

The purpose of this note is to present a proof of Hall's matching theorem (also called marriage theorem) which I have not encountered elsewhere in literature - what yet does not mean that it is necessarily new.

We refer to Hall's theorem in the following form:

> **Theorem 1 (Hall).** Let n be a positive integer. Let Γ be a bipartite graph whose set of vertices consists of n blue vertices B_1, B_2, ..., B_n and n green vertices G_1, G_2, ..., G_n. Then, the graph Γ has a perfect matching if and only if every subset $J \subseteq \{1, 2, ..., n\}$ satisfies $\left| \bigcup\limits_{i \in J} \mathcal{N}(G_i) \right| \geq |J|$.

Some notations used in this theorem require *explanations*:

- A *bipartite* graph is a (simple, non-directed) graph with each vertex colored either green or blue such that every edge of the graph connects a blue vertex and a green vertex.
- A *perfect matching* of the bipartite graph Γ means a permutation ϕ of the set $\{1, 2, ..., n\}$ such that for every $j \in \{1, 2, ..., n\}$, the vertex B_j is connected to the vertex $G_{\phi(j)}$.
- The number of elements of a finite set X is denoted by $|X|$.
- Finally, if A is a vertex of our graph Γ, then a *neighbour* of A means any other vertex of Γ which is connected to A by an edge. We denote by $\mathcal{N}(A)$ the set of all neighbours of A.

Proofs of Theorem 1 abound in literature - see, e. g., Chapter 11 of [1], Theorem 12.2 in [2], or Theorem 2.1.2 in [3]. Here we are going to present a proof which is longer than most of these, but applies an idea apparently new, and potentially interesting for further study.

Proof of Theorem 1. In order to show Theorem 1, we have to verify two assertions:

Assertion 1. If the graph Γ has a perfect matching, then every subset $J \subseteq \{1, 2, ..., n\}$ satisfies $\left| \bigcup\limits_{i \in J} \mathcal{N}(G_i) \right| \geq |J|$.

Assertion 2. If the graph Γ has no perfect matching, then there exists a subset $J \subseteq \{1, 2, ..., n\}$ which does *not* satisfy $\left| \bigcup\limits_{i \in J} \mathcal{N}(G_i) \right| \geq |J|$.

We begin with the easy *proof of Assertion 1:* Assume that the graph Γ has a perfect matching, i. e. there exists a permutation ϕ of the set $\{1, 2, ..., n\}$ such that for every $j \in \{1, 2, ..., n\}$, the vertex B_j is connected to the vertex $G_{\phi(j)}$. This means that $B_j \in \mathcal{N}\left(G_{\phi(j)}\right)$ for every $j \in \{1, 2, ..., n\}$. Now, since ϕ is a permutation, it has an inverse permutation - that is, a permutation ψ of the set $\{1, 2, ..., n\}$ satisfying $\psi \circ \phi = \phi \circ \psi = \mathrm{id}$. Then, since $B_j \in \mathcal{N}\left(G_{\phi(j)}\right)$ for every $j \in \{1, 2, ..., n\}$, we must also have $B_{\psi(j)} \in \mathcal{N}\left(G_{\phi(\psi(j))}\right)$ for every $j \in \{1, 2, ..., n\}$. Since $\phi(\psi(j)) = (\phi \circ \psi)(j) = \mathrm{id}(j) = j$, this becomes $B_{\psi(j)} \in \mathcal{N}(G_j)$. So we have $B_{\psi(j)} \in \mathcal{N}(G_j)$ for every $j \in \{1, 2, ..., n\}$.

Consider any subset $J \subseteq \{1, 2, ..., n\}$. Then, $\bigcup\limits_{i \in J} \mathcal{N}(G_i) \supseteq \left\{B_{\psi(j)} \mid j \in J\right\}$ (because for every $j \in J$, we have $B_{\psi(j)} \in \mathcal{N}(G_j) \subseteq \bigcup\limits_{i \in J} \mathcal{N}(G_i)$). Thus, $\left|\bigcup\limits_{i \in J} \mathcal{N}(G_i)\right| \geq \left|\left\{B_{\psi(j)} \mid j \in J\right\}\right|$. But $\left|\left\{B_{\psi(j)} \mid j \in J\right\}\right| = |J|$ (because any two different $j \in J$ yield two different $B_{\psi(j)}$, since ψ is a permutation!). Hence, $\left|\bigcup\limits_{i \in J} \mathcal{N}(G_i)\right| \geq |J|$.

We have thus shown that every subset $J \subseteq \{1, 2, ..., n\}$ satisfies $\left|\bigcup\limits_{i \in J} \mathcal{N}(G_i)\right| \geq |J|$. This proves Assertion 1.

The interesting part is the *proof of Assertion 2.* Before we come to this proof, we define some notations concerning matrices:

- For a matrix A, we denote by $A\left[\begin{array}{c} j \\ i \end{array}\right]$ the entry in the j-th column and the i-th row of A. [This is usually denoted by A_{ij}.]

- Let A be a matrix with u rows and v columns. Let $j_1, j_2, ..., j_k$ be some pairwise distinct integers from the set $\{1, 2, ..., v\}$, and let $i_1, i_2, ..., i_l$ be some pairwise distinct integers from the set $\{1, 2, ..., u\}$. Then, we denote by $A\left[\dfrac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_l}\right]$ the matrix with l rows and k columns which is defined as follows: For any integers $p \in \{1, 2, ..., l\}$ and $q \in \{1, 2, ..., k\}$, we have $\left(A\left[\dfrac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_l}\right]\right)\left[\begin{array}{c} q \\ p \end{array}\right] = A\left[\begin{array}{c} j_q \\ i_p \end{array}\right]$.

  Informally speaking, $A\left[\dfrac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_l}\right]$ is the matrix formed by the intersections of the columns numbered $j_1, j_2, ..., j_k$ with the rows numbered $i_1, i_2, ..., i_l$ of the matrix A, but the order of these columns and rows depends on the order of the integers $j_1, j_2, ..., j_k$ and the order of the integers $i_1, i_2, ..., i_l$.

  Such a matrix $A\left[\dfrac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_l}\right]$ is called a *minor* of the matrix A.

  Examples:

$$\left(\begin{array}{cccc} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \end{array}\right)\left[\frac{2,4}{1,3}\right] = \left(\begin{array}{cc} b & d \\ b'' & d'' \end{array}\right);$$

$$\left(\begin{array}{ccc} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{array}\right)\left[\frac{3,1}{1,2,3}\right] = \left(\begin{array}{cc} c & a \\ c' & a' \\ c'' & a'' \end{array}\right).$$

Note that, thus, for any matrix A, the matrix $A\left[\frac{j}{i}\right]$ is the 1×1 matrix consisting of the only element $A\left[\begin{array}{c} j \\ i \end{array}\right]$.

- If m is a positive integer, and $r\in\{1,2,...,m\}$, then the notation j_1, j_2, ..., $\widehat{j_r}$, ..., j_m is going to mean "the numbers j_1, j_2, ..., j_m with the number j_r left out" (i. e. "the numbers j_1, j_2, ..., j_{r-2}, j_{r-1}, j_{r+1}, j_{r+2}, ..., j_m ").

We will make use of a method of computing determinants called *developing a determinant along a row.* This method states that for any $k\times k$ matrix U and any $s\in\{1,2,...,k\}$, we have

$$\det U=\sum_{r=1}^{k}(-1)^{s+r}\cdot U\left[\begin{array}{c} r \\ s \end{array}\right]\cdot\det\left(U\left[\frac{1,2,...,\widehat{r},...,k}{1,2,...,\widehat{s},...,k}\right]\right). \qquad (1)$$

Let us also introduce two basic notations:

- For two sets U and V, the assertion $U\subset V$ will mean that U is a proper subset of V (that is, $U\subseteq V$ and $U\neq V$).
- Let M be a set, and let $\mathcal{A}(X)$ be an assertion defined for every subset X of M. Then, a subset S of M will be called *a minimal subset of M satisfying $\mathcal{A}$* if and only if the assertion $\mathcal{A}(S)$ is true, while the assertion $\mathcal{A}(T)$ is wrong for every proper subset T of S.

An easy fact:

Lemma 2. Let M be a finite set. Let $\mathcal{A}(X)$ is an assertion defined for every subset X of M. Assume that the assertion $\mathcal{A}(M)$ is valid. Then, there exists a minimal subset of M satisfying $\mathcal{A}$.

Proof of Lemma 2. A nonnegative integer t will be called *nice* if there exists a subset N of M satisfying $\mathcal{A}(N)$ and $|N|=t$. The set of nice nonnegative integers is non-empty (in fact, the nonnegative integer $|M|$ is nice, since the subset M of M satisfies $\mathcal{A}(M)$ and $|M|=|M|$). Hence, there exists a smallest nice nonnegative integer. Let k be the smallest nice nonnegative integer.

Then, there exists a subset S of M satisfying $\mathcal{A}(S)$ and $|S|=k$. For any proper subset T of S, we have $|T|<|S|=k$, so that $\mathcal{A}(T)$ is wrong (because if $\mathcal{A}(T)$ would be true, then $|T|$ would be a nice nonnegative integer, but since $|T|<k$ this would contradict to the definition of k as the smallest nice nonnegative integer). Thus, S is a minimal subset of M satisfying $\mathcal{A}$. Hence, the existence of a minimal subset of M satisfying $\mathcal{A}$ is proven, i. e. the proof of Lemma 2 is complete.

Now, to our proof of Assertion 2. We assume that the graph Γ has no perfect matching. This means, there is no permutation ϕ of the set $\{1,2,...,n\}$ such that for every $j\in\{1,2,...,n\}$, the vertex B_j is connected to the vertex $G_{\phi(j)}$. In other words,

(*) for every permutation π of the set $\{1,2,...,n\}$, there exists some $i\in\{1,2,...,n\}$ such that the vertex B_i is not connected to the vertex $G_{\pi(i)}$.

In order to prove Assertion 2, we have to find a subset $J \subseteq \{1, 2, ..., n\}$ which does *not* satisfy $\left|\bigcup_{i \in J} \mathcal{N}(G_i)\right| \geq |J|$.

Let K be an arbitrary field (for instance, $\mathbb{Q}$). Let L be the field of all rational functions of n^2 indeterminates $X_{1,1}$, $X_{1,2}$, ..., $X_{n,n}$ (one indeterminate $X_{i,j}$ for each pair $(i, j) \in \{1, 2, ..., n\}^2$) over K.

Then, $L = K(X_{1,1}, X_{1,2}, ..., X_{n,n})$.

We define a matrix $S \in \mathrm{M}_n(L)$ by setting

$$S\left[\begin{array}{c} j \\ i \end{array}\right] = \begin{cases} X_{i,j}, & \text{if } G_j \in \mathcal{N}(B_i); \\ 0, & \text{if } G_j \notin \mathcal{N}(B_i) \end{cases} \qquad \text{for any two } i \text{ and } j \text{ from the set } \{1, 2, ..., n\}.$$

This matrix S stores all information about the bipartite graph Γ in it: For any blue vertex B_i and any green vertex G_j, we can tell whether B_i and G_j are connected from the entry $S\left[\begin{array}{c} j \\ i \end{array}\right]$ of this matrix (in fact, the vertices B_i and G_j are connected if and only if $S\left[\begin{array}{c} j \\ i \end{array}\right] \neq 0$).

By the definition of the determinant as a sum over permutations, we have

$$\det S = \sum_{\pi \in S_n} \operatorname{sign} \pi \cdot \prod_{i=1}^{n} S\left[\begin{array}{c} \pi(i) \\ i \end{array}\right].$$

Now, for every permutation $\pi \in S_n$, the product $\prod_{i=1}^{n} S\left[\begin{array}{c} \pi(i) \\ i \end{array}\right]$ equals 0 (because this product always has one of its factors equal to 0 - in fact, according to **(*)**, there exists some $i \in \{1, 2, ..., n\}$ such that the vertex B_i is not connected to the vertex $G_{\pi(i)}$; this means that, for this i, we have $G_{\pi(i)} \notin \mathcal{N}(B_i)$, so that $S\left[\begin{array}{c} \pi(i) \\ i \end{array}\right] = \begin{cases} X_{i,\pi(i)}, & \text{if } G_{\pi(i)} \in \mathcal{N}(B_i); \\ 0, & \text{if } G_{\pi(i)} \notin \mathcal{N}(B_i) \end{cases} = 0$). Hence,

$$\det S = \sum_{\pi \in S_n} \operatorname{sign} \pi \cdot \prod_{i=1}^{n} S\left[\begin{array}{c} \pi(i) \\ i \end{array}\right] = \sum_{\pi \in S_n} \operatorname{sign} \pi \cdot 0 = 0.$$

Thus, the matrix S is non-invertible. Thus, the columns of the matrix S are linearly dependent.

For any subset $T = \{t_1, t_2, ..., t_k\}$ of $\{1, 2, ..., n\}$, let $\mathcal{A}(T)$ be the assertion that the columns of the matrix S numbered t_1, t_2, ..., t_k are linearly dependent (hereby, when we write $T = \{t_1, t_2, ..., t_k\}$, we assume that the numbers t_1, t_2, ..., t_k are pairwisely distinct). The set $\{1, 2, ..., n\}$ is finite, and the assertion $\mathcal{A}(\{1, 2, ..., n\})$ is valid (since the columns of the matrix S numbered 1, 2, ..., n are linearly dependent, because these are all columns of the matrix S, and as we know these are linearly dependent). Hence, by Lemma 2, there exists a minimal subset of $\{1, 2, ..., n\}$ satisfying $\mathcal{A}$. Let $\{j_1, j_2, ..., j_k\}$ be such a minimal subset (with the numbers j_1, j_2, ..., j_k being pairwisely distinct). Then, the assertion $\mathcal{A}(\{j_1, j_2, ..., j_k\})$ is true, while the assertion $\mathcal{A}(T)$ is wrong for every proper subset T of $\{j_1, j_2, ..., j_k\}$.

Thus, the k columns of the matrix S numbered j_1, j_2, ..., j_k are linearly dependent (because $\mathcal{A}(\{j_1, j_2, ..., j_k\})$ holds), but for every $u \in \{1, 2, ..., k\}$, the $k-1$ columns of the matrix S numbered j_1, j_2, ..., $\widehat{j_u}$, ..., j_k are linearly independent (because $\left\{j_1, j_2, ..., \widehat{j_u}, ..., j_k\right\}$ is a proper subset of $\{j_1, j_2, ..., j_k\}$, so that $\mathcal{A}\left(\left\{j_1, j_2, ..., \widehat{j_u}, ..., j_k\right\}\right)$ is wrong, i. e. the columns of the matrix S numbered j_1, j_2, ..., $\widehat{j_u}$, ..., j_k are linearly independent).

In other words, the k columns of the matrix $S\left[\dfrac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$ are linearly dependent, but for every $u \in \{1, 2, ..., k\}$, the $k-1$ columns of the matrix $S\left[\dfrac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$ numbered 1, 2, ..., $\widehat{u}$, ..., k are linearly independent. Hence, the matrix $S\left[\dfrac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$ has rank $k-1$ (in fact, its rank is $< k$, because its k columns are linearly dependent, but on the other hand its rank is $\geq k-1$, because it has $k-1$ linearly independent columns (in fact, for any $u \in \{1, 2, ..., k\}$, its $k-1$ columns numbered 1, 2, ..., $\widehat{u}$, ..., k are linearly independent)).

Since the matrix $S\left[\dfrac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$ has rank $k-1$, it has $k-1$ linearly independent rows, and every row of this matrix is a linear combination of these $k-1$ rows.

So let the rows numbered i_1, i_2, ..., i_{k-1} be $k-1$ linearly independent rows of the matrix $S\left[\dfrac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$. Then, every row of the matrix $S\left[\dfrac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$ is a linear combination of these $k-1$ rows i_1, i_2, ..., i_{k-1}. In other words, for every $i \in \{1, 2, ..., n\}$, there exist elements $\alpha_{i,\ 1}$, $\alpha_{i,\ 2}$, ..., $\alpha_{i,\ k-1}$ of L such that the i-th row of the matrix $S\left[\dfrac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$ is the sum of $\alpha_{i,\ v}$ times the i_v-th row of this matrix over all $v \in \{1, 2, ..., k-1\}$. This means that

$$S\left[\begin{array}{c} j_u \\ i \end{array}\right] = \sum_{v=1}^{k-1} \alpha_{i,\ v} S\left[\begin{array}{c} j_u \\ i_v \end{array}\right]$$

for every $u \in \{1, 2, ..., k\}$.

Now, we will show that for each $r \in \{1, 2, ..., k\}$, we have $\det\left(S\left[\dfrac{j_1, j_2, ..., \widehat{j_r}, ..., j_k}{i_1, i_2, ..., i_{k-1}}\right]\right) \neq 0$. In fact, assume that this is not the case. Then, there exists some $r \in \{1, 2, ..., k\}$ such that $\det\left(S\left[\dfrac{j_1, j_2, ..., \widehat{j_r}, ..., j_k}{i_1, i_2, ..., i_{k-1}}\right]\right) = 0$. Hence, for this r, the $k-1$ columns of the matrix $S\left[\dfrac{j_1, j_2, ..., \widehat{j_r}, ..., j_k}{i_1, i_2, ..., i_{k-1}}\right]$ are linearly dependent. In other words, the columns j_1, j_2, ..., $\widehat{j_r}$, ..., j_k of the matrix $S\left[\dfrac{1, 2, ..., n}{i_1, i_2, ..., i_{k-1}}\right]$ are linearly dependent. This means that there exist elements β_1, β_2, ..., $\widehat{\beta_r}$, ..., β_k of L which are not all equal to 0 such that the sum of β_u times the j_u-th column of the matrix $S\left[\dfrac{1, 2, ..., n}{i_1, i_2, ..., i_{k-1}}\right]$ over all

$u \in \{1, 2, ..., \widehat{r}, ..., k\}$ equals 0. Equivalently,

$$\sum_{1 \leq u \leq k;\ u \neq r} \beta_u \cdot S\left[\begin{array}{c} j_u \\ i_v \end{array}\right] = 0$$

for each $v \in \{1, 2, ..., k-1\}$. Then, for every $i \in \{1, 2, ..., n\}$, using the relation $S\left[\begin{array}{c} j_u \\ i \end{array}\right] = \sum\limits_{v=1}^{k-1} \alpha_{i,\ v} S\left[\begin{array}{c} j_u \\ i_v \end{array}\right]$ which holds for every $u \in \{1, 2, ..., k\}$, we obtain

$$\begin{aligned} \sum_{1 \leq u \leq k;\ u \neq r} \beta_u \cdot S\left[\begin{array}{c} j_u \\ i \end{array}\right] &= \sum_{1 \leq u \leq k;\ u \neq r} \beta_u \cdot \sum_{v=1}^{k-1} \alpha_{i,\ v} S\left[\begin{array}{c} j_u \\ i_v \end{array}\right] \\ &= \sum_{v=1}^{k-1} \alpha_{i,\ v} \underbrace{\sum_{1 \leq u \leq k;\ u \neq r} \beta_u \cdot S\left[\begin{array}{c} j_u \\ i_v \end{array}\right]}_{=0} = 0. \end{aligned}$$

In other words, the sum of β_u times the j_u-th column of the matrix S over all $u \in \{1, 2, ..., \widehat{r}, ..., k\}$ equals 0. Since the elements $\beta_1, \beta_2, ..., \widehat{\beta_r}, ..., \beta_k$ are not all equal to 0, this yields that the columns of the matrix S numbered $j_1, j_2, ..., \widehat{j_r}, ..., j_k$ are linearly dependent. But this contradicts to the fact that for every $u \in \{1, 2, ..., k\}$, the $k-1$ columns of the matrix S numbered $j_1, j_2, ..., \widehat{j_u}, ..., j_k$ are linearly independent. This contradiction yields that our assumption was wrong. Thus, we have proven that for each $r \in \{1, 2, ..., k\}$, we have

$$\det\left(S\left[\frac{j_1, j_2, ..., \widehat{j_r}, ..., j_k}{i_1, i_2, ..., i_{k-1}}\right]\right) \neq 0. \qquad (2)$$

Now let $J = \{j_1, j_2, ..., j_k\}$. Then, we are going to prove that $\bigcup\limits_{i \in J} \mathcal{N}(G_i) \subseteq \{B_{i_1}, B_{i_2}, ..., B_{i_{k-1}}\}$.

In fact, we are going to prove this by contradiction: Assume that $\bigcup\limits_{i \in J} \mathcal{N}(G_i) \subseteq \{B_{i_1}, B_{i_2}, ..., B_{i_{k-1}}\}$ does not hold. Then, there exists a vertex T of the graph Γ which lies in $\bigcup\limits_{i \in J} \mathcal{N}(G_i)$ but not in $\{B_{i_1}, B_{i_2}, ..., B_{i_{k-1}}\}$.

From $T \in \bigcup\limits_{i \in J} \mathcal{N}(G_i)$, it follows that there exists some $j \in J$ with $T \in \mathcal{N}(G_j)$. Thus, T is a blue vertex of the graph Γ (in fact, since G_j is a green vertex, all neighbours of G_j are blue vertices (since the graph Γ is bipartite), so that T is a blue vertex because $T \in \mathcal{N}(G_j)$). Thus, $T = B_{\widetilde{i}}$ for some $\widetilde{i} \in \{1, 2, ..., n\}$. But since $T \notin \{B_{i_1}, B_{i_2}, ..., B_{i_{k-1}}\}$, we must have $\widetilde{i} \notin \{i_1, i_2, ..., i_{k-1}\}$.

Besides, since $j \in J = \{j_1, j_2, ..., j_k\}$, there exists an $q \in \{1, 2, ..., k\}$ such that $j = j_q$. Since $T = B_{\widetilde{i}}$ and $j = j_q$, the relation $T \in \mathcal{N}(G_j)$ becomes $B_{\widetilde{i}} \in \mathcal{N}(G_{j_q})$. Thus, the vertices $B_{\widetilde{i}}$ and G_{j_q} are connected, so that $G_{j_q} \in \mathcal{N}(B_{\widetilde{i}})$. Hence,

$$S\left[\begin{array}{c} j_q \\ \widetilde{i} \end{array}\right] = \left\{\begin{array}{c} X_{\widetilde{i}, j_q}, \text{ if } G_{j_q} \in \mathcal{N}(B_{\widetilde{i}}); \\ 0, \text{ if } G_{j_q} \notin \mathcal{N}(B_{\widetilde{i}}) \end{array}\right. = X_{\widetilde{i}, j_q}.$$

Since the numbers $i_1, i_2, ..., i_{k-1}$ are pairwisely distinct (because the rows of the matrix $S\left[\frac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$ numbered $i_1, i_2, ..., i_{k-1}$ are linearly independent) and we have

$\widetilde{i} \notin \{i_1, i_2, ..., i_{k-1}\}$, we can conclude that the numbers i_1, i_2, ..., i_{k-1}, $\widetilde{i}$ are pairwisely distinct. Now consider the square matrix $S\left[\dfrac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]$. This matrix is a $k \times k$ matrix, but its rank is $\leq k-1$ (in fact, this matrix is a minor of the matrix $S\left[\dfrac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$, so its rank must be $\leq$ to the rank of $S\left[\dfrac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$, but the rank of $S\left[\dfrac{j_1, j_2, ..., j_k}{1, 2, ..., n}\right]$ is known to be $k-1$). Hence, the determinant of this matrix must be 0; that is,

$$\det\left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right) = 0. \tag{3}$$

But on the other hand, by developing the determinant of the matrix $S\left[\dfrac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]$ along its last (that is, its k-th) row (i. e., by applying the formula (1) to $U = S\left[\dfrac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]$ and $s = k$), we obtain

$$\begin{aligned}
&\det\left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right) \\
&= \sum_{r=1}^{k} (-1)^{k+r} \cdot \left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right)\left[\begin{array}{c} r \\ k \end{array}\right] \cdot \det\left(\left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right)\left[\frac{1, 2, ..., \widehat{r}, ..., k}{1, 2, ..., \widehat{k}, ..., k}\right]\right)
\end{aligned}$$

(hereby, of course, 1, 2, ..., $\widehat{k}$, ..., k is just a complicated notation for 1, 2, ..., $k-1$). Once we take note that

$$\left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right)\left[\begin{array}{c} r \\ k \end{array}\right] = S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right]$$

and

$$\left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right)\left[\frac{1, 2, ..., \widehat{r}, ..., k}{1, 2, ..., \widehat{k}, ..., k}\right] = S\left[\frac{j_1, j_2, ..., \widehat{j_r}, ..., j_k}{i_1, i_2, ..., i_{k-1}}\right],$$

this monstrous equation simplifies to

$$\begin{aligned}
&\det\left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right) \\
&= \sum_{r=1}^{k} (-1)^{k+r} \cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right] \cdot \det\left(S\left[\frac{j_1, j_2, ..., \widehat{j_r}, ..., j_k}{i_1, i_2, ..., i_{k-1}}\right]\right).
\end{aligned} \tag{4}$$

Denote $d_r = \det\left(S\left[\dfrac{j_1, j_2, ..., \widehat{j_r}, ..., j_k}{i_1, i_2, ..., i_{k-1}}\right]\right)$ for every $r \in \{1, 2, ..., k\}$. Then, (2) yields $d_r \neq 0$ for every $r \in \{1, 2, ..., k\}$, while (4) transforms into

$$\det\left(S\left[\frac{j_1, j_2, ..., j_k}{i_1, i_2, ..., i_{k-1}, \widetilde{i}}\right]\right) = \sum_{r=1}^{k} (-1)^{k+r} \cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right] \cdot d_r.$$

Comparing this with (3), we obtain

$$0=\sum_{r=1}^{k}(-1)^{k+r}\cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right]\cdot d_r.$$

This rewrites as

$$0=\sum_{1\leq r\leq k;\ r\neq q}(-1)^{k+r}\cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right]\cdot d_r+(-1)^{k+q}\cdot S\left[\begin{array}{c} j_q \\ \widetilde{i} \end{array}\right]\cdot d_q.$$

Hence,

$$(-1)^{k+q}\cdot S\left[\begin{array}{c} j_q \\ \widetilde{i} \end{array}\right]\cdot d_q=-\sum_{1\leq r\leq k;\ r\neq q}(-1)^{k+r}\cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right]\cdot d_r.$$

Since $(-1)^{k+q}\neq 0$ and $d_q\neq 0$ (because $d_r\neq 0$ for every $r\in\{1,2,...,k\}$), we have $(-1)^{k+q}\cdot d_q\neq 0$, so that we can divide this equation by $(-1)^{k+q}\cdot d_q$, and obtain

$$S\left[\begin{array}{c} j_q \\ \widetilde{i} \end{array}\right]=\frac{-\sum\limits_{1\leq r\leq k;\ r\neq q}(-1)^{k+r}\cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right]\cdot d_r}{(-1)^{k+q}\cdot d_q}. \tag{5}$$

Now we will prove that

$$\frac{-\sum\limits_{1\leq r\leq k;\ r\neq q}(-1)^{k+r}\cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right]\cdot d_r}{(-1)^{k+q}\cdot d_q}\in K\left(X_{1,1},X_{1,2},...,\widehat{X_{\widetilde{i},j_q}},...,X_{n,n}\right).$$

In fact, first it is obvious that

$$\text{for every } r\in\{1,2,...,k\}\,,\ \text{we have } (-1)^{k+r}\in K\left(X_{1,1},X_{1,2},...,\widehat{X_{\widetilde{i},j_q}},...,X_{n,n}\right). \tag{6}$$

Particularly this yields

$$(-1)^{k+q}\in K\left(X_{1,1},X_{1,2},...,\widehat{X_{\widetilde{i},j_q}},...,X_{n,n}\right). \tag{7}$$

Now we are going to show that

$$\text{for every } r\in\{1,2,...,k\}\,,\ \text{we have } d_r\in K\left(X_{1,1},X_{1,2},...,\widehat{X_{\widetilde{i},j_q}},...,X_{n,n}\right). \tag{8}$$

Proof of the relation (8): For every $x\in\{1,2,...,k\}$ and every $y\in\{1,2,...,k-1\}$, we have $S\left[\begin{array}{c} j_x \\ i_y \end{array}\right]\in K\left(X_{1,1},X_{1,2},...,\widehat{X_{\widetilde{i},j_q}},...,X_{n,n}\right)$, because $S\left[\begin{array}{c} j_x \\ i_y \end{array}\right]=\begin{cases} X_{i_y,j_x}, \text{ if } G_{j_x}\in\mathcal{N}\left(B_{i_y}\right); \\ 0, \text{ if } G_{j_x}\notin\mathcal{N}\left(B_{i_y}\right)\end{cases}$ and $X_{i_y,j_x}\in K\left(X_{1,1},X_{1,2},...,\widehat{X_{\widetilde{i},j_q}},...,X_{n,n}\right)$ (the latter because $\widetilde{i}\notin\{i_1,i_2,...,i_{k-1}\}$ yields $i_y\neq\widetilde{i}$). Hence, all entries of the matrix $S\left[\begin{array}{c} j_1,j_2,...,\widehat{j_r},...,j_k \\ \hline i_1,i_2,...,i_{k-1} \end{array}\right]$ lie in the field $K\left(X_{1,1},X_{1,2},...,\widehat{X_{\widetilde{i},j_q}},...,X_{n,n}\right)$ (because all these entries have the form $S\left[\begin{array}{c} j_x \\ i_y \end{array}\right]$ for

$x \in \{1, 2, ..., k\}$ and $y \in \{1, 2, ..., k-1\}$). Thus, the determinant of this matrix, being a polynomial of its entries, must also lie in the field $K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right)$. In other words, $\det\left(S\left[\dfrac{j_1, j_2, ..., \widehat{j_r}, ..., j_k}{i_1, i_2, ..., i_{k-1}}\right]\right) \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right)$. Since $d_r = \det\left(S\left[\dfrac{j_1, j_2, ..., \widehat{j_r}, ..., j_k}{i_1, i_2, ..., i_{k-1}}\right]\right)$, this becomes $d_r \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right)$, and thus (8) is proven.

Applying (8) to $r = q$, we get

$$d_q \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right). \tag{9}$$

Finally,

$$\begin{aligned} &\text{for every } r \in \{1, 2, ..., k\} \text{ with } r \neq q, \text{ we have} \\ &S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right] \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right). \end{aligned} \tag{10}$$

This is because $S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right] = \begin{cases} X_{\widetilde{i},j_r}, & \text{if } G_{j_r} \in \mathcal{N}\left(B_{\widetilde{i}}\right); \\ 0, & \text{if } G_{j_r} \notin \mathcal{N}\left(B_{\widetilde{i}}\right) \end{cases}$ and $X_{\widetilde{i},j_r} \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right)$ (the latter because $r \neq q$ yields $j_r \neq j_q$).

From (6), (7), (8), (9) and (10) together, it follows that

$$\frac{-\sum\limits_{1\leq r\leq k;\ r\neq q} (-1)^{k+r} \cdot S\left[\begin{array}{c} j_r \\ \widetilde{i} \end{array}\right] \cdot d_r}{(-1)^{k+q} \cdot d_q} \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right).$$

Using (5), this transforms into $S\left[\begin{array}{c} j_q \\ \widetilde{i} \end{array}\right] \in K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right)$. But this is wrong, because we know that $S\left[\begin{array}{c} j_q \\ \widetilde{i} \end{array}\right] = X_{\widetilde{i},j_q} \notin K\left(X_{1,1}, X_{1,2}, ..., \widehat{X_{\widetilde{i},j_q}}, ..., X_{n,n}\right)$. Hence, we have obtained a contradiction.

This contradiction shows that our assumption was wrong. Hence, we do have $\bigcup\limits_{i\in J} \mathcal{N}(G_i) \subseteq \left\{B_{i_1}, B_{i_2}, ..., B_{i_{k-1}}\right\}$. Thus, $\left|\bigcup\limits_{i\in J} \mathcal{N}(G_i)\right| \leq \left|\left\{B_{i_1}, B_{i_2}, ..., B_{i_{k-1}}\right\}\right|$. But $\left|\left\{B_{i_1}, B_{i_2}, ..., B_{i_{k-1}}\right\}\right| = k-1$ because the vertices B_{i_1}, B_{i_2}, ..., $B_{i_{k-1}}$ are pairwisely distinct (since the numbers i_1, i_2, ..., i_{k-1} are pairwisely distinct). But $|J| = |\{j_1, j_2, ..., j_k\}| = k$. Hence,

$$\left|\bigcup_{i\in J} \mathcal{N}(G_i)\right| \leq \left|\left\{B_{i_1}, B_{i_2}, ..., B_{i_{k-1}}\right\}\right| = k-1 < k = |J|.$$

Thus, the subset $J \subseteq \{1, 2, ..., n\}$ does *not* satisfy $\left|\bigcup\limits_{i\in J} \mathcal{N}(G_i)\right| \geq |J|$. This proves Assertion 2.

As both Assertions 1 and 2 are shown now, the proof of Theorem 1 is complete.

References

[1] George Pólya, Robert E. Tarjan, Donald R. Woods, *Notes on Introductory Combinatorics*, Boston/Basel/Stuttgart 1983.

[2] L. Lovász, K. Vesztergombi, *Discrete Mathematics*, lecture notes, 1999.
http://www.cs.tau.ac.il/~odedr/teaching
/discrete_math_fall_2005/dmbook.pdf

[3] Reinhard Diestel, *Graph Theory*, 3rd Edition, Heidelberg 2005.
http://www.math.uni-hamburg.de/home/diestel/books/graph.theory/