An unexpected application of the Gergonne-Euler theorem
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This note is based on a problem from the IMO longlist 1976 proposed by Great Britain (GBR 2 in [1]). When I first saw that problem, I spent a longer time solving it, and the solution obtained was rather nonstandard for an olympiad geometry problem. Before we state the problem, four conventions are appropriate:

- The point of intersection of two lines g and h will be denoted by $g \cap h$ in the following.

- The parallel to a line g through a point P will be denoted by $\operatorname{para}(P;\ g)$.

- We will use directed lengths (also known as signed lengths). Hereby, the directed length of a segment PQ will be denoted by $\overline{PQ}$ (of course, this directed length is only defined if the line through the points P and Q is directed, but we can work with *ratios* of directed lengths on non-directed lines as well). The usual, non-directed distance between two points P and Q will be denoted by PQ.

- We work in the projective plane with the Euclidean structure on its Euclidean component. This means that we work as one usually works in Euclidean geometry, but a formulation of the kind "the three lines concur at one point" will also include the case that these three lines concur at one infinite point, i. e. are all parallel to each other. We will consider such cases as limiting cases, i. e. we won't pay particular attention to them even if they require a modification of our arguments.

Now we are ready to formulate the assertion of the IMO longlist problem (Fig. 1):

Theorem 1. Let ABC and $A'B'C'$ be two triangles on a plane. Denote

$$X = BC \cap B'C'; \qquad Y = CA \cap C'A'; \qquad Z = AB \cap A'B';$$
$$X' = \operatorname{para}(A;\ BC) \cap \operatorname{para}(A';\ B'C'); \qquad Y' = \operatorname{para}(B;\ CA) \cap \operatorname{para}(B';\ C'A');$$
$$Z' = \operatorname{para}(C;\ AB) \cap \operatorname{para}(C';\ A'B').$$

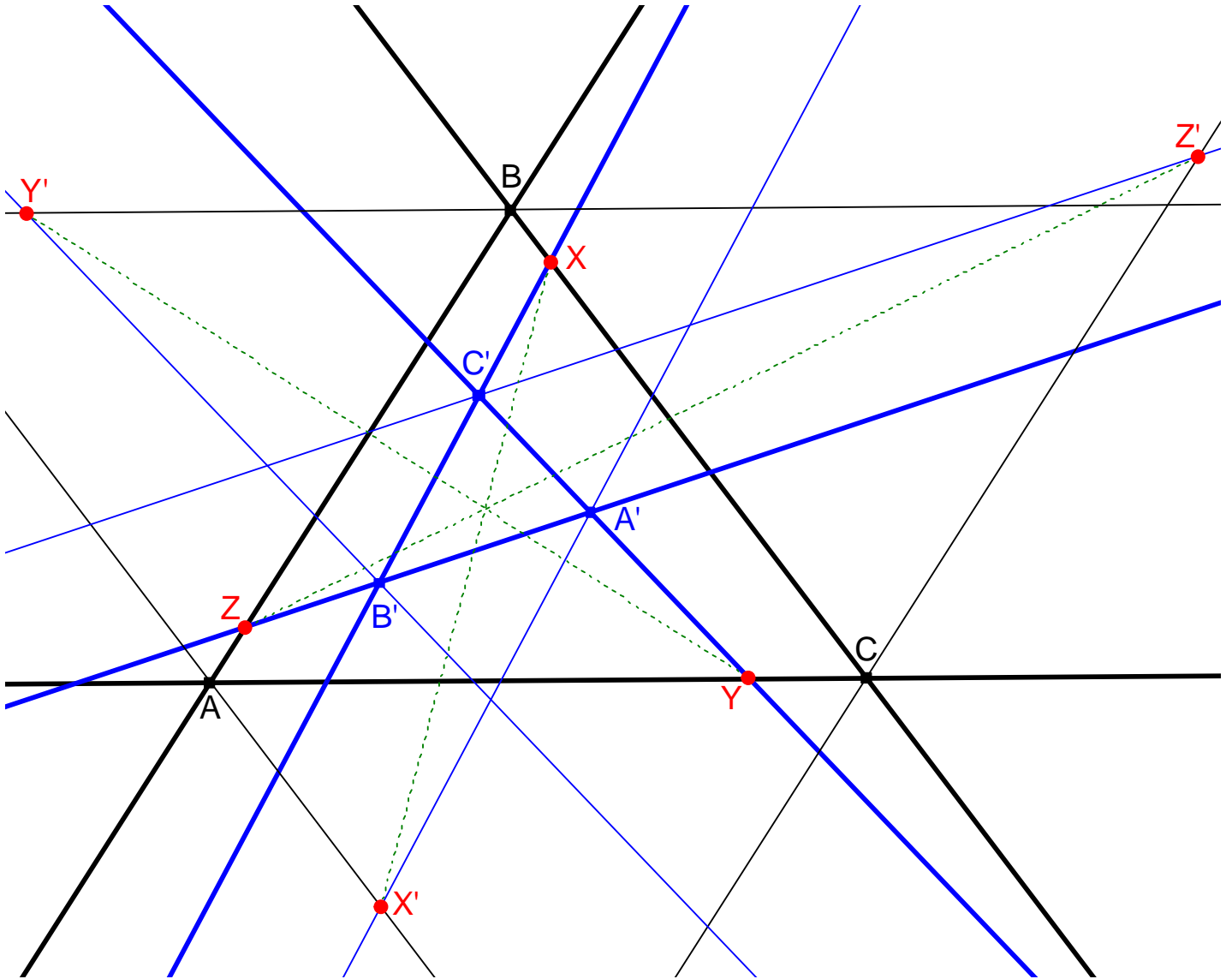Then, the lines XX', YY', ZZ' concur at one point.

Fig. 1

The solution is based on the following fact (Fig. 2):

Theorem 2, the Gergonne-Euler theorem. Let ABC be a triangle, and P a point in its plane. The lines AP, BP, CP intersect the lines BC, CA, AB at the points A_1, B_1, C_1. Then,

$$\frac{\overline{PA_1}}{\overline{AA_1}} + \frac{\overline{PB_1}}{\overline{BB_1}} + \frac{\overline{PC_1}}{\overline{CC_1}} = 1.$$

Remark. The assertion of this theorem can be equivalently stated in the form $\frac{\overline{AP}}{\overline{AA_1}} + \frac{\overline{BP}}{\overline{BB_1}} + \frac{\overline{CP}}{\overline{CC_1}} = 2$ as well as in the form $\frac{\overline{AP}}{\overline{PA_1}} \cdot \frac{\overline{BP}}{\overline{PB_1}} \cdot \frac{\overline{CP}}{\overline{PC_1}} = \frac{\overline{AP}}{\overline{PA_1}} + \frac{\overline{BP}}{\overline{PB_1}} + \frac{\overline{CP}}{\overline{PC_1}} + 2.$
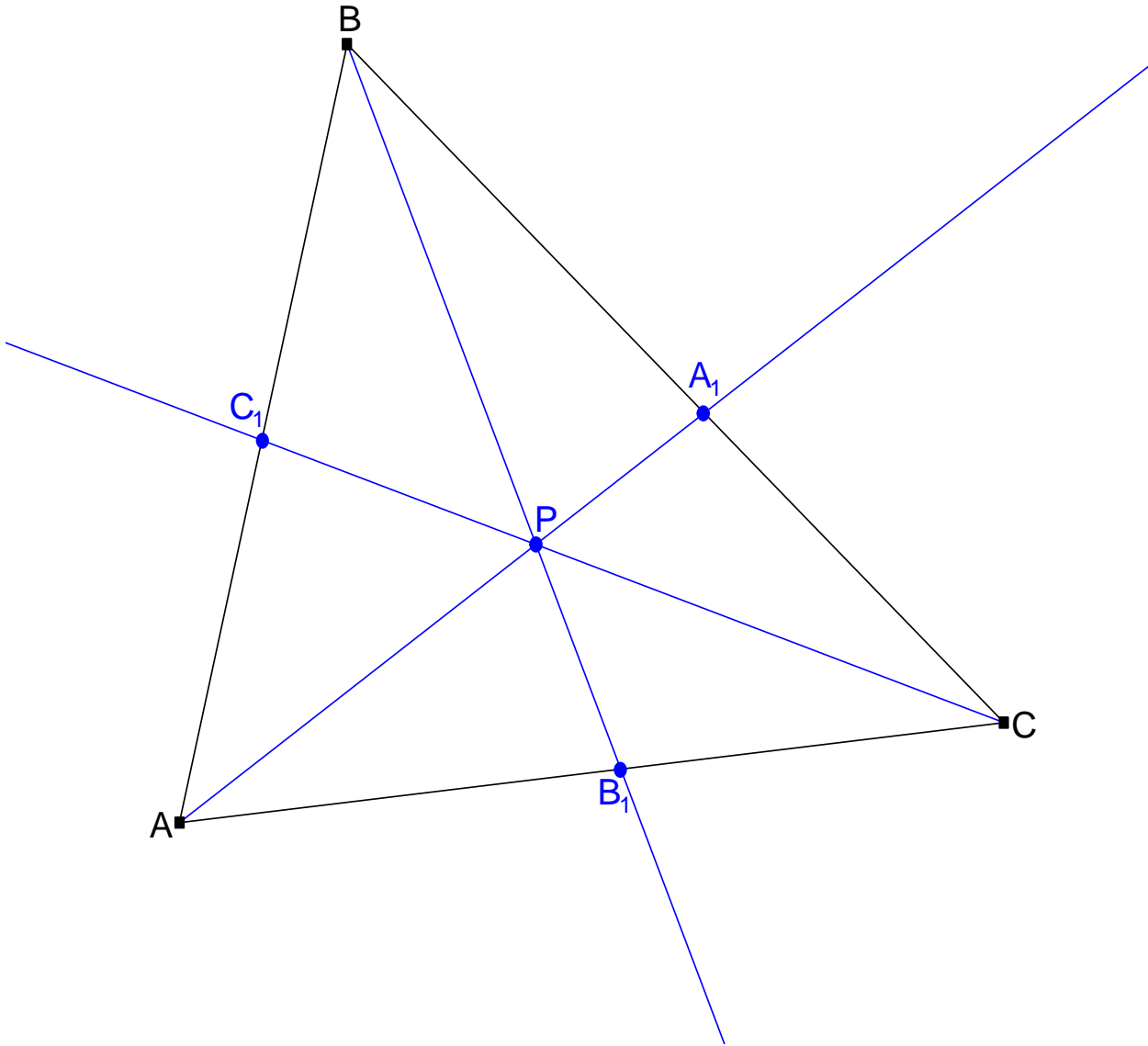Proving the equivalence is a simple calculation exercise.

Fig. 2

Proof of Theorem 2. (See Fig. 3.) Without loss of generality, we consider only the case when the point P lies inside the triangle ABC. Let H_b and P_b be the orthogonal projections of the points B and P on the line CA. Then, $BH_b \perp CA$ and $PP_b \perp CA$ together yield $BH_b \parallel PP_b$, and thus, by Thales, we have $\dfrac{PB_1}{BB_1} = \dfrac{PP_b}{BH_b}$.
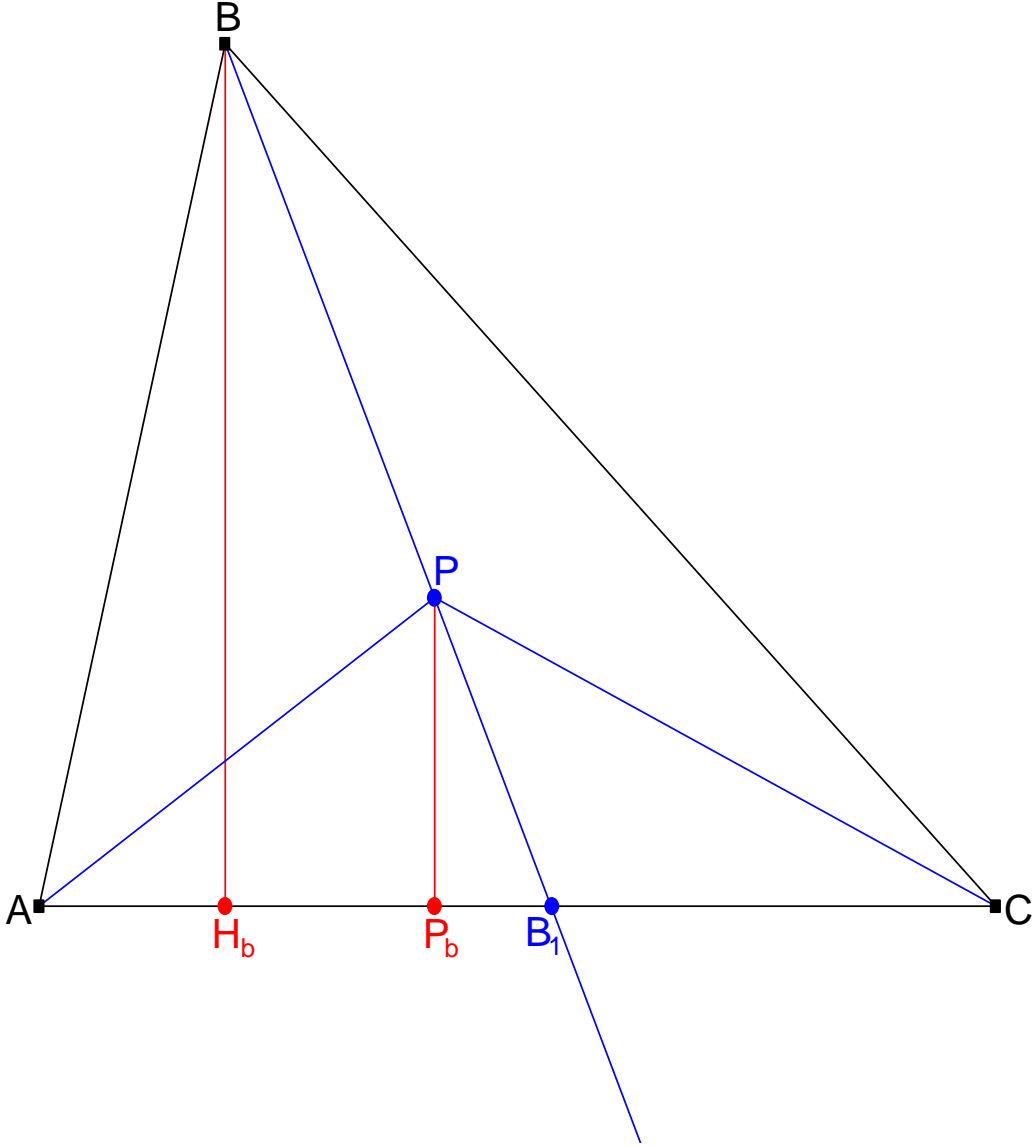
Fig. 3

Now we denote by $|P_1P_2P_3|$ the (non-directed) area of an arbitrary triangle $P_1P_2P_3$. Since the area of a triangle equals $\dfrac{1}{2} \cdot$ sidelength $\cdot$ corresponding altitude, we have $|ABC| = \dfrac{1}{2} \cdot CA \cdot BH_b$ (since triangle ABC has CA as a side and BH_b as the corresponding altitude) and $|CPA| = \dfrac{1}{2} \cdot CA \cdot PP_b$ (since triangle CPA has CA as a side and PP_b as the corresponding altitude). Thus, $\dfrac{|CPA|}{|ABC|} = \dfrac{\dfrac{1}{2} \cdot CA \cdot PP_b}{\dfrac{1}{2} \cdot CA \cdot BH_b} = \dfrac{PP_b}{BH_b}$.

Comparing this to $\dfrac{PB_1}{BB_1} = \dfrac{PP_b}{BH_b}$, we get $\dfrac{PB_1}{BB_1} = \dfrac{|CPA|}{|ABC|}$. Similarly, $\dfrac{PC_1}{CC_1} = \dfrac{|APB|}{|ABC|}$

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and $\dfrac{PA_1}{AA_1} = \dfrac{|BPC|}{|ABC|}$. Hence,

$$\frac{PA_1}{AA_1} + \frac{PB_1}{BB_1} + \frac{PC_1}{CC_1} = \frac{|BPC|}{|ABC|} + \frac{|CPA|}{|ABC|} + \frac{|APB|}{|ABC|} = \frac{|BPC| + |CPA| + |APB|}{|ABC|} = \frac{|ABC|}{|ABC|} = 1.$$

Now, $\dfrac{PA_1}{AA_1} = \dfrac{\overline{PA_1}}{\overline{AA_1}}$, $\dfrac{PB_1}{BB_1} = \dfrac{\overline{PB_1}}{\overline{BB_1}}$ and $\dfrac{PC_1}{CC_1} = \dfrac{\overline{PC_1}}{\overline{CC_1}}$ (since P lies inside triangle ABC),

and thus this becomes $\dfrac{\overline{PA_1}}{\overline{AA_1}} + \dfrac{\overline{PB_1}}{\overline{BB_1}} + \dfrac{\overline{PC_1}}{\overline{CC_1}} = 1$. This proves Theorem 2.
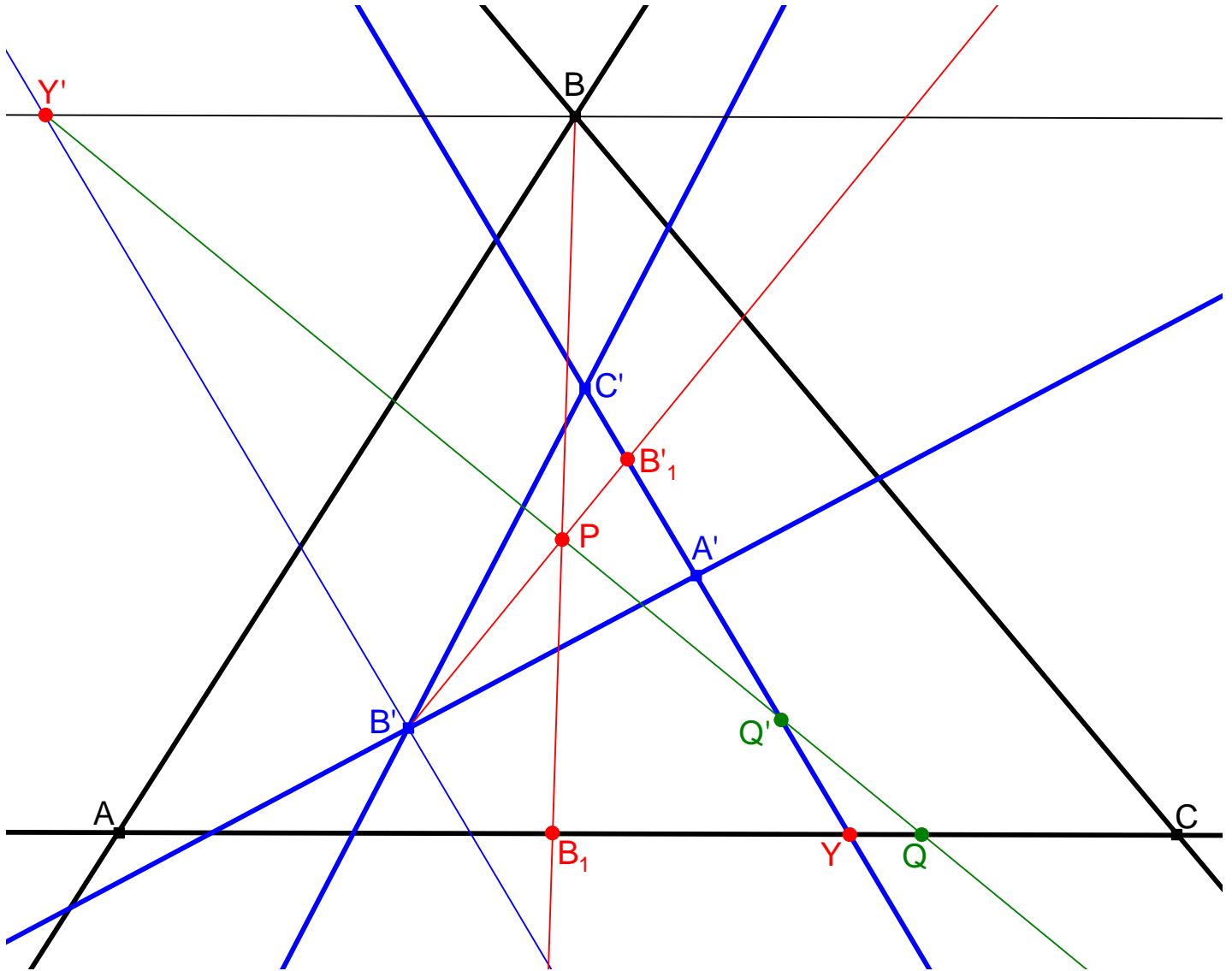

Fig. 4

Next we establish a lemma (see Fig. 4 for Lemma 3 **b)**):

> **Lemma 3.** In the configuration of Theorem 1, let P be an arbitrary point in the plane. The lines AP, BP, CP intersect the lines BC, CA, AB at the

points A_1, B_1, C_1. The lines $A'P$, $B'P$, $C'P$ intersect the lines $B'C'$, $C'A'$, $A'B'$ at the points A'_1, B'_1, C'_1. Then:

a) The point P lies on the line XX' if and only if $\dfrac{\overline{PA_1}}{\overline{AA_1}} = \dfrac{\overline{PA'_1}}{\overline{A'A'_1}}$.

b) The point P lies on the line YY' if and only if $\dfrac{\overline{PB_1}}{\overline{BB_1}} = \dfrac{\overline{PB'_1}}{\overline{B'B'_1}}$.

c) The point P lies on the line ZZ' if and only if $\dfrac{\overline{PC_1}}{\overline{CC_1}} = \dfrac{\overline{PC'_1}}{\overline{C'C'_1}}$.

Proof of Lemma 3. (See Fig. 4.) Let $Q = PY' \cap CA$ and $Q' = PY' \cap C'A'$. Since $Y' \in \text{para}\,(B;\ CA)$, we have $BY' \parallel CA$, and thus, by Thales, $\dfrac{\overline{PB_1}}{\overline{BB_1}} = \dfrac{\overline{PQ}}{\overline{Y'Q}}$. Since $Y' \in \text{para}\,(B';\ C'A')$, we have $B'Y' \parallel C'A'$, and thus, by Thales, $\dfrac{\overline{PB'_1}}{\overline{B'B'_1}} = \dfrac{\overline{PQ'}}{\overline{Y'Q'}}$.

Now, we construct a chain of obviously equivalent assertions:
(The point P lies on the line YY')
$\Longleftrightarrow$ (The line PY' passes through the point Y)
$\Longleftrightarrow$ (The line PY' passes through the point $CA \cap C'A'$)
$\Longleftrightarrow$ (The line PY' intersects the lines CA and $C'A'$ at the same point)
$\Longleftrightarrow$ $(PY' \cap CA = PY' \cap C'A') \Longleftrightarrow (Q = Q')$
$\Longleftrightarrow \left(\dfrac{\overline{PQ}}{\overline{Y'Q}} = \dfrac{\overline{PQ'}}{\overline{Y'Q'}}\right) \Longleftrightarrow \left(\dfrac{\overline{PB_1}}{\overline{BB_1}} = \dfrac{\overline{PB'_1}}{\overline{B'B'_1}}\right),$

where the last equivalence is due to $\dfrac{\overline{PB_1}}{\overline{BB_1}} = \dfrac{\overline{PQ}}{\overline{Y'Q}}$ and $\dfrac{\overline{PB'_1}}{\overline{B'B'_1}} = \dfrac{\overline{PQ'}}{\overline{Y'Q'}}$. This chain proves Lemma 3 **b)**. Lemma 3 **a)** and **c)** are proven in an analogous way, and thus the proof of Lemma 3 is complete.

Combining the above, we now complete the *proof of Theorem 1:* Denote by P the point of intersection of the lines XX' and YY'. Let A_1, B_1, C_1 be the points of intersection of the lines AP, BP, CP with the lines BC, CA, AB, respectively. Let A'_1, B'_1, C'_1 be the points of intersection of the lines $A'P$, $B'P$, $C'P$ with the lines $B'C'$, $C'A'$, $A'B'$, respectively.

Since P lies on XX', Lemma 3 **a)** yields $\dfrac{\overline{PA_1}}{\overline{AA_1}} = \dfrac{\overline{PA'_1}}{\overline{A'A'_1}}$. Since P lies on YY', Lemma 3 **b)** yields $\dfrac{\overline{PB_1}}{\overline{BB_1}} = \dfrac{\overline{PB'_1}}{\overline{B'B'_1}}$. Now, Theorem 2, applied to the triangle ABC and the point P (with the lines AP, BP, CP intersecting the lines BC, CA, AB at A_1, B_1, C_1), yields $\dfrac{\overline{PA_1}}{\overline{AA_1}} + \dfrac{\overline{PB_1}}{\overline{BB_1}} + \dfrac{\overline{PC_1}}{\overline{CC_1}} = 1$. Using $\dfrac{\overline{PA_1}}{\overline{AA_1}} = \dfrac{\overline{PA'_1}}{\overline{A'A'_1}}$ and $\dfrac{\overline{PB_1}}{\overline{BB_1}} = \dfrac{\overline{PB'_1}}{\overline{B'B'_1}}$, this transforms into $\dfrac{\overline{PA'_1}}{\overline{A'A'_1}} + \dfrac{\overline{PB'_1}}{\overline{B'B'_1}} + \dfrac{\overline{PC_1}}{\overline{CC_1}} = 1$.

On the other hand, Theorem 2, applied to the triangle $A'B'C'$ and the point P (with the lines $A'P$, $B'P$, $C'P$ intersecting the lines $B'C'$, $C'A'$, $A'B'$ at A'_1, B'_1, C'_1),

yields $\dfrac{\overline{PA'_1}}{\overline{A'A'_1}} + \dfrac{\overline{PB'_1}}{\overline{B'B'_1}} + \dfrac{\overline{PC'_1}}{\overline{C'C'_1}} = 1$. Comparing this with $\dfrac{\overline{PA'_1}}{\overline{A'A'_1}} + \dfrac{\overline{PB'_1}}{\overline{B'B'_1}} + \dfrac{\overline{PC_1}}{\overline{CC_1}} = 1$, we

get $\dfrac{\overline{PC_1}}{\overline{CC_1}} = \dfrac{\overline{PC'_1}}{\overline{C'C'_1}}$. According to Lemma 3 **c)**, this shows that P lies on ZZ'.

Thus, the lines XX', YY' and ZZ' concur at one point - namely, at the point P. This proves Theorem 1.

We note in passing that Theorem 1 can be proven in a different way as well:

There exists an affine transformation of the plane which maps the points A, B, C to the points A', B', C'. If this transformation has a fixed point, then it can be shown that this fixed point lies on the lines XX', YY', ZZ'. If this transformation has no fixed points, then one can see that the lines XX', YY', ZZ' are all parallel to each other. The details of this proof are left to the reader.

As a further application of the Gergonne-Euler theorem, we can show (see Fig. 2 again):

> **Theorem 4, the van Aubel theorem.** Let ABC be a triangle, and let P be a point in its plane. The lines AP, BP, CP intersect the lines BC, CA, AB at the points A_1, B_1, C_1. Then,
>
> $$\frac{\overline{AP}}{\overline{PA_1}} = \frac{\overline{AC_1}}{\overline{C_1B}} + \frac{\overline{AB_1}}{\overline{B_1C}}; \tag{1}$$
>
> $$\frac{\overline{BP}}{\overline{PB_1}} = \frac{\overline{BA_1}}{\overline{A_1C}} + \frac{\overline{BC_1}}{\overline{C_1A}}; \tag{2}$$
>
> $$\frac{\overline{CP}}{\overline{PC_1}} = \frac{\overline{CB_1}}{\overline{B_1A}} + \frac{\overline{CA_1}}{\overline{A_1B}}. \tag{3}$$

This result is classical and easy to prove using the Thales theorem and auxiliary points. Here we will derive it from Theorem 2:

Consider the triangle PBC and the point A in its plane. The lines PA, BA, CA intersect the lines BC, CP, PB at the points A_1, C_1, B_1. Thus, the equation (1) of Theorem 2 yields

$$\frac{\overline{AA_1}}{\overline{PA_1}} + \frac{\overline{AC_1}}{\overline{BC_1}} + \frac{\overline{AB_1}}{\overline{CB_1}} = 1, \qquad \text{so that}$$

$$\frac{\overline{AA_1}}{\overline{PA_1}} - 1 = -\left(\frac{\overline{AC_1}}{\overline{BC_1}} + \frac{\overline{AB_1}}{\overline{CB_1}} \right).$$

But $\dfrac{\overline{AA_1}}{\overline{PA_1}} - 1 = \dfrac{\overline{AA_1} - \overline{PA_1}}{\overline{PA_1}} = \dfrac{\overline{AP}}{\overline{PA_1}}$ and

$$-\left(\frac{\overline{AC_1}}{\overline{BC_1}} + \frac{\overline{AB_1}}{\overline{CB_1}} \right) = \left(-\frac{\overline{AC_1}}{\overline{BC_1}} \right) + \left(-\frac{\overline{AB_1}}{\overline{CB_1}} \right) = \frac{\overline{AC_1}}{\overline{C_1B}} + \frac{\overline{AB_1}}{\overline{B_1C}}.$$

Hence, this becomes $\dfrac{\overline{AP}}{\overline{PA_1}} = \dfrac{\overline{AC_1}}{\overline{C_1B}} + \dfrac{\overline{AB_1}}{\overline{B_1C}}$. This proves (4), and similarly (5) and (6) can be established.

We have thus deduced Theorem 4 from Theorem 2. Similarly, by the way, we could have deduced Theorem 2 from Theorem 4 as well.

References

[1] Dušan Djukić, Vladimir Janković, Ivan Matić, Nikola Petrović, *The IMO Compendium*, Springer 2006.