An adventitious angle problem concerning $\sqrt{2}$ and $\frac{\pi}{7}$ / Darij Grinberg

The purpose of this note is to give two solutions of the following problem (Fig. 1):

Let ABC be an isosceles triangle with $AB = AC$ and $BC = 1$. Let P be a point on the side AB of this triangle which satisfies $AP = 1$.

Prove that $CP = \sqrt{2}$ holds if and only if $\measuredangle CAB = \frac{\pi}{7}$.

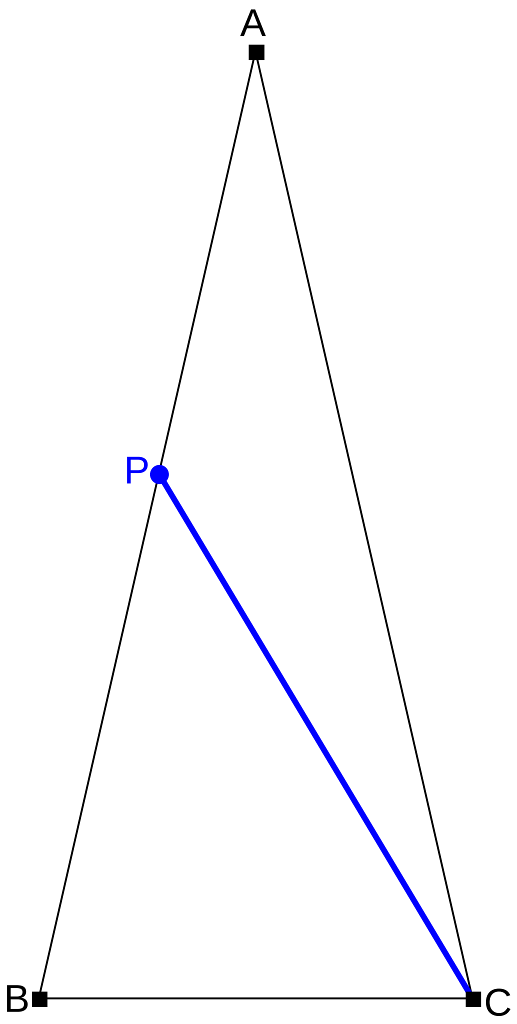


Fig. 1

It is not hard to solve this problem using trigonometry or complex numbers (see, e. g., the MathLinks discussion

`http://www.mathlinks.ro/Forum/viewtopic.php?t=22849`

for the direction $\measuredangle CAB = \frac{\pi}{7} \Rightarrow CP = \sqrt{2}$).

Here, we will present two synthetic solutions of the problem; the first one was given (for the direction $CP = \sqrt{2} \Rightarrow \measuredangle CAB = \frac{\pi}{7}$) by Stefan V. (a pseudonym), the second one is apparently original.

First solution (**Stefan V**.):

Before solving the problem, we recall two facts on parallelograms. The first one is a pretty well-known formula:

Lemma 1. Let $ABCD$ be a parallelogram. Then, $AC^2 + BD^2 = 2 \bullet (AB^2 + BC^2)$.

In other words, the sum of the squares of the diagonals of a parallelogram is equal to the double sum of the squares of two adjacent sides. (See Fig. 2.)

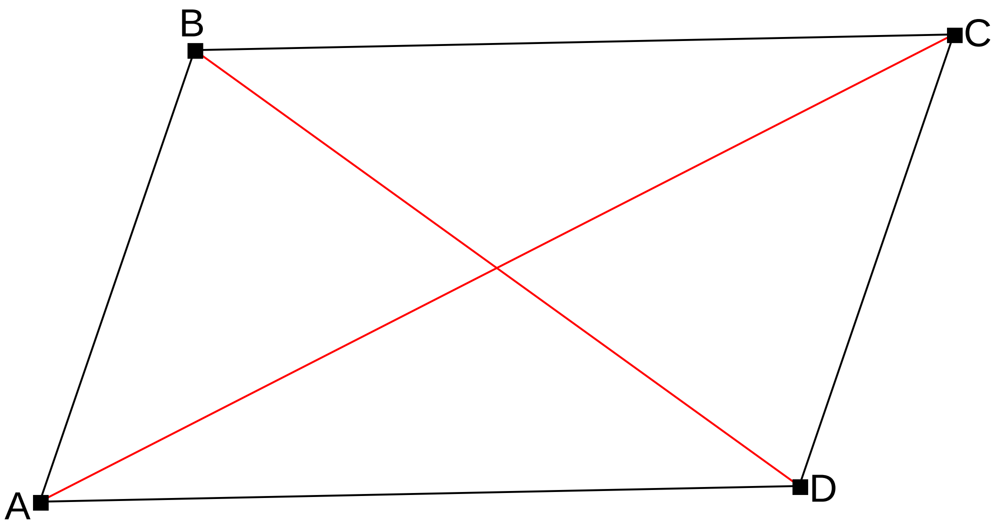


Fig. 2

Lemma 1 is most easily proven using vectors and their scalar products: Since *ABCD* is a parallelogram, we have $\overrightarrow{CD} = \overrightarrow{BA}$, or, equivalently, $\overrightarrow{CD} = -\overrightarrow{AB}$. Thus $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BC} - \overrightarrow{AB}$, and hence

$$\begin{aligned} AC^2 + BD^2 = \overrightarrow{AC}^2 + \overrightarrow{BD}^2 &= \left(\overrightarrow{AB} + \overrightarrow{BC}\right)^2 + \left(\overrightarrow{BC} - \overrightarrow{AB}\right)^2 \\ &= \left(\overrightarrow{AB}^2 + 2 \bullet \overrightarrow{AB} \bullet \overrightarrow{BC} + \overrightarrow{BC}^2\right) + \left(\overrightarrow{BC}^2 - 2 \bullet \overrightarrow{BC} \bullet \overrightarrow{AB} + \overrightarrow{AB}^2\right) \\ &= 2 \bullet \left(\overrightarrow{AB}^2 + \overrightarrow{BC}^2\right) = 2 \bullet (AB^2 + BC^2), \end{aligned}$$

so Lemma 1 is proven. Note that Lemma 1 is more known in the form $AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2$, which is trivially equivalent to $AC^2 + BD^2 = 2 \bullet (AB^2 + BC^2)$ since $AB = CD$ and $BC = DA$ (because *ABCD* is a parallelogram).

The next property of parallelograms applied below will be:

Lemma 2. Let *ABCD* be a parallelogram. If $AC = \sqrt{2} \bullet AB$, then $BD = \sqrt{2} \bullet BC$.

In other words, if in a parallelogram, a diagonal is $\sqrt{2}$ times as long as a side, then the other diagonal is $\sqrt{2}$ times as long as the other side. (See Fig. 3.)

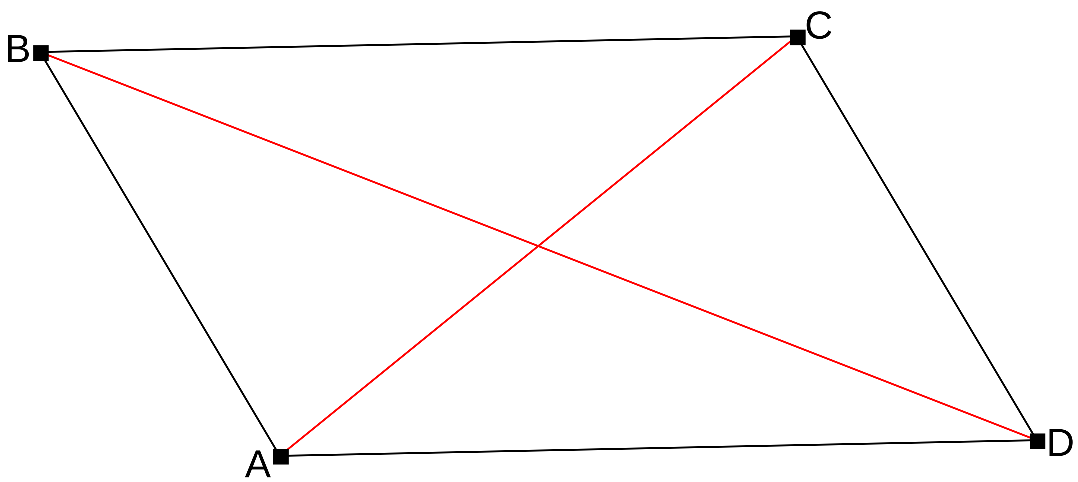


Fig. 3

In fact, Lemma 2 is a trivial corollary of Lemma 1: If $AC = \sqrt{2} \bullet AB$, then $AC^2 = 2 \bullet AB^2$; subtracting this from the equation $AC^2 + BD^2 = 2 \bullet (AB^2 + BC^2)$ which holds by Lemma 1, we obtain $BD^2 = 2 \bullet BC^2$, so that $BD = \sqrt{2} \bullet BC$, and Lemma 2 is proven.

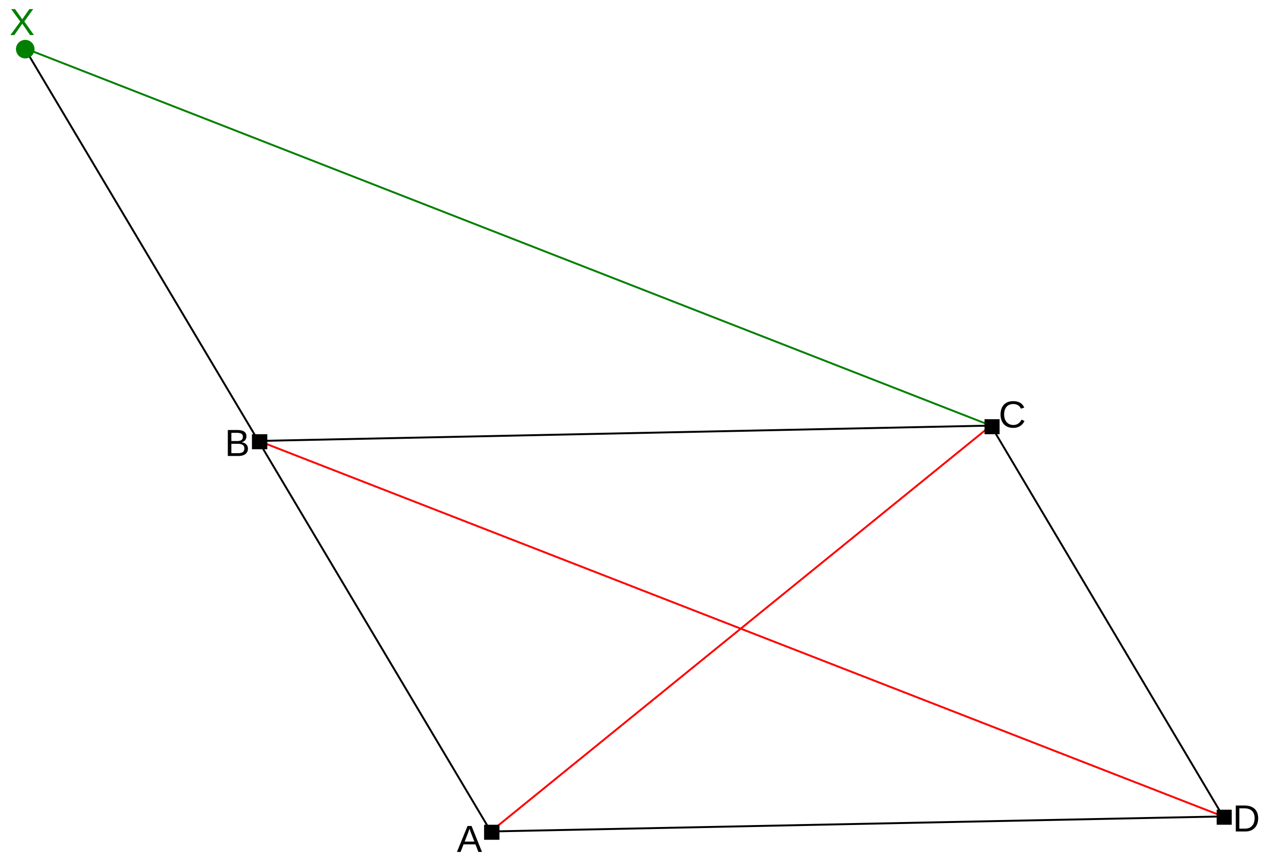


Fig. 4

There is also an alternative proof of Lemma 2 using similar triangles (Fig. 4): Let X be the reflection of the point A in the point B. Then, $BX = AB$. On the other hand, $AB = DC$, since $ABCD$ is a parallelogram. Thus, $BX = DC$. Together with $BX \parallel DC$ (what follows from $AB \parallel DC$, what is because $ABCD$ is a parallelogram), this yields that the quadrilateral $BDCX$ is a parallelogram, so that $XC = BD$.

Now, we supposed that $AC = \sqrt{2} \bullet AB$, so that $AC^2 = 2 \bullet AB^2$. In other words, $\frac{AC}{AB} = \frac{2 \bullet AB}{AC}$. But $BX = AB$ yields $2 \bullet AB = AB + BX = AX$, so this becomes $\frac{AC}{AB} = \frac{AX}{AC}$. Since we also trivially have $\measuredangle CAB = \measuredangle XAC$, we can conclude that the triangles CAB and XAC are similar. Thus, $\frac{XC}{CB} = \frac{AC}{AB}$. Since $XC = BD$ and $AC = \sqrt{2} \bullet AB$, this becomes $\frac{BD}{CB} = \frac{\sqrt{2} \bullet AB}{AB} = \sqrt{2}$; hence, $BD = \sqrt{2} \bullet CB = \sqrt{2} \bullet BC$. This again proves Lemma 2.

Now we come to the actual solution of the problem:

In order to solve the problem, we have to prove two assertions:

Assertion 1: If $CP = \sqrt{2}$, then $\measuredangle CAB = \frac{\pi}{7}$.

Assertion 2: If $\measuredangle CAB = \frac{\pi}{7}$, then $CP = \sqrt{2}$.

Before we verify these two assertions, we perform some observations independent of the validity of $CP = \sqrt{2}$ and $\measuredangle CAB = \frac{\pi}{7}$.

(See Fig. 5.) Let the parallel to the line AB through the point C meet the parallels to the lines BC and AC through the point P at the points S and R.

We have $CS \parallel AB$, or, equivalently, $CS \parallel BP$, and we have $PS \parallel BC$; thus, the quadrilateral $BCSP$ is a parallelogram. Thus, $CS = BP$. On the other hand, we have $CR \parallel AB$, or, equivalently, $CR \parallel AP$, and we have $PR \parallel AC$; thus, the quadrilateral $ACRP$ is a parallelogram. This yields $CR = AP$. Hence, $RS = CR + CS = AP + BP = AB$. Together with $RS \parallel AB$ this implies that the quadrilateral $ABRS$ is a parallelogram.

Let $\measuredangle CAB = \alpha$. Since triangle ABC is isosceles, its base angle $\measuredangle ABC$ then equals

$$\triangle ABC = \frac{\pi - \triangle CAB}{2} = \frac{\pi - \alpha}{2}.$$

Since $CR \parallel AB,$ we have $\triangle BCR = \triangle ABC,$ so that $\triangle BCR = \frac{\pi - \alpha}{2}.$

Now $CR = AP = 1 = BC;$ thus, the triangle BCR is isosceles, so its base angle is

$$\triangle CBR = \frac{\pi - \triangle BCR}{2} = \frac{\pi - \frac{\pi - \alpha}{2}}{2} = \frac{\left(\frac{\pi + \alpha}{2}\right)}{2} = \frac{\pi + \alpha}{4}.$$

Hence,

$$\triangle PBR = \triangle ABC + \triangle CBR = \frac{\pi - \alpha}{2} + \frac{\pi + \alpha}{4} = \frac{2(\pi - \alpha) + (\pi + \alpha)}{4} = \frac{3\pi - \alpha}{4}.$$

Now, $PR \parallel AC$ implies $\triangle RPB = \triangle CAB,$ so that $\triangle RPB = \alpha$. Thus, the sum of angles in triangle PBR yields

$$\triangle BRP = \pi - \triangle PBR - \triangle RPB = \pi - \frac{3\pi - \alpha}{4} - \alpha = \frac{\pi + \alpha}{4} - \alpha = \frac{\pi - 3\alpha}{4}.$$

Now, we have $BP = BR$ if and only if the triangle PBR is isosceles with base $PR;$ this holds if and only if $\triangle RPB = \triangle BRP,$ i. e. if $\alpha = \frac{\pi - 3\alpha}{4};$ but this is obviously equivalent to $4\alpha = \pi - 3\alpha,$ hence to $\alpha = \frac{\pi}{7}$. So we have shown that $BP = BR$ holds if and only if $\alpha = \frac{\pi}{7}$.

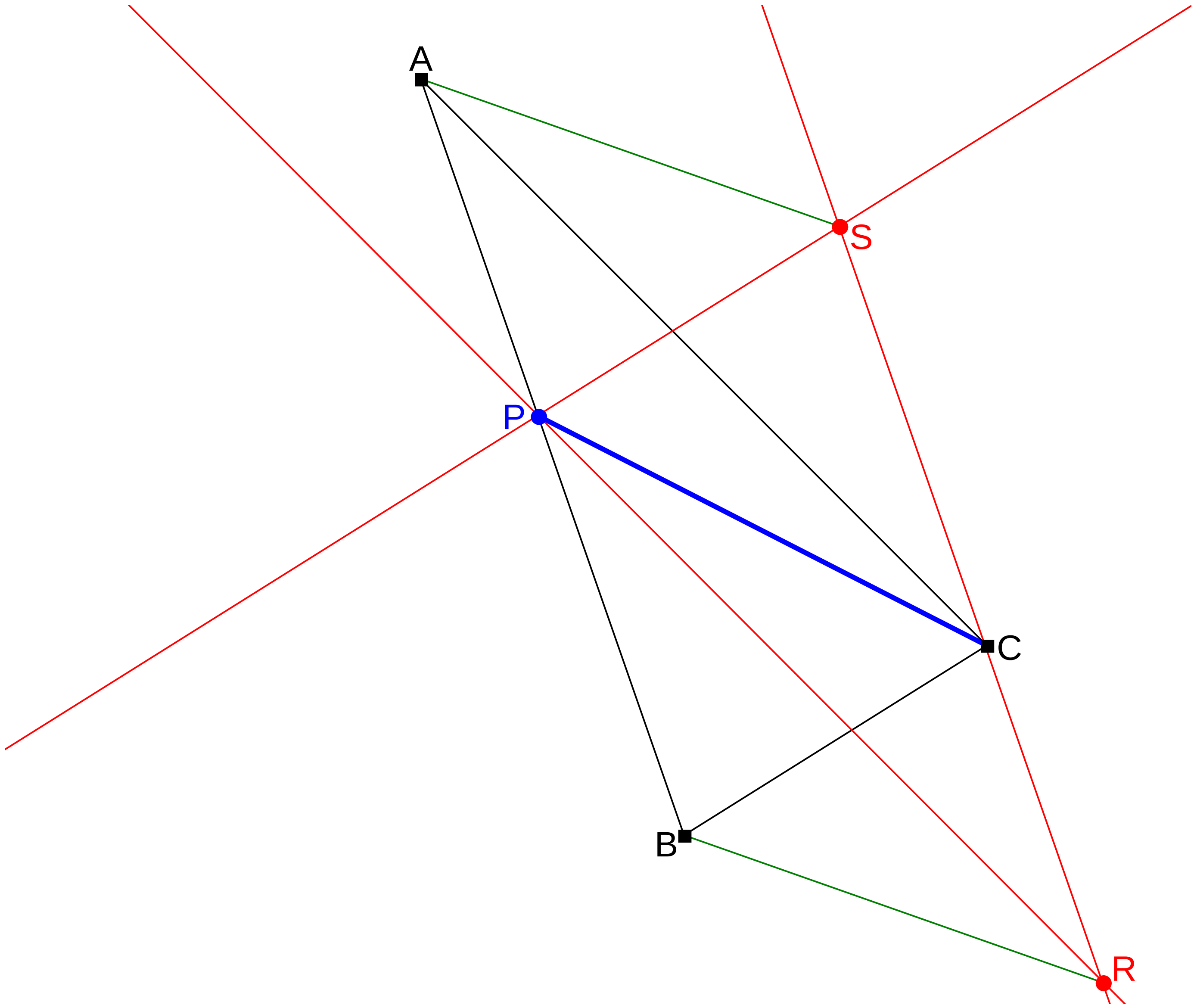


Fig. 5

Now, we will prove the Assertions 1 and 2. We start with the *proof of Assertion* 1:

Assume that $CP = \sqrt{2}$. Since $AP = 1$, this rewrites as $CP = \sqrt{2} \bullet AP$. By Lemma 2, applied to the parallelogram $ACRP$, this entails $AR = \sqrt{2} \bullet AC$. Since $AB = AC$, this rewrites as $AR = \sqrt{2} \bullet AB$. According to Lemma 2, applied to the parallelogram $ABRS$, this leads to $BS = \sqrt{2} \bullet BR$. But since $BC = 1$, we can rewrite the equation $CP = \sqrt{2}$ in the form $CP = \sqrt{2} \bullet BC$ as well, and thus, from Lemma 2, applied to the parallelogram $BCSP$, we conclude that $BS = \sqrt{2} \bullet BP$. Comparing this with $BS = \sqrt{2} \bullet BR$, we get $BP = BR$. As showed above, this is equivalent to $\alpha = \frac{\pi}{7}$, i. e. to $\measuredangle CAB = \frac{\pi}{7}$, and thus Assertion 1 is proven.

More difficult is the *proof of Assertion* 2:

Assume that $\measuredangle CAB = \frac{\pi}{7}$. In other words, $\alpha = \frac{\pi}{7}$. According to the above, this yields $BP = BR$.

Application of Lemma 1 to the parallelogram $BCSP$ yields $BS^2 + CP^2 = 2 \bullet (BC^2 + BP^2)$, what, in view of $BC^2 = 1^2 = 1$, becomes $BS^2 + CP^2 = 2 \bullet (1 + BP^2)$.

Application of Lemma 1 to the parallelogram $ACRP$ yields $CP^2 + AR^2 = 2 \bullet (AC^2 + AP^2)$, what, in view of $AC = AB$ and $AP^2 = 1^2 = 1$, becomes $CP^2 + AR^2 = 2 \bullet (AB^2 + 1)$.

Application of Lemma 1 to the parallelogram $ABRS$ yields $AR^2 + BS^2 = 2 \bullet (AB^2 + BR^2)$, what, in view of $BP = BR$, becomes $AR^2 + BS^2 = 2 \bullet (AB^2 + BP^2)$.

Thus,

$$CP^2 = \frac{2 \bullet CP^2}{2} = \frac{(BS^2 + CP^2) + (CP^2 + AR^2) - (AR^2 + BS^2)}{2}$$
$$= \frac{2 \bullet (1 + BP^2) + 2 \bullet (AB^2 + 1) - 2 \bullet (AB^2 + BP^2)}{2} = 2,$$

so that $CP = \sqrt{2}$. Thus, Assertion 2 is proven, and the solution of the problem is complete.

Second solution:

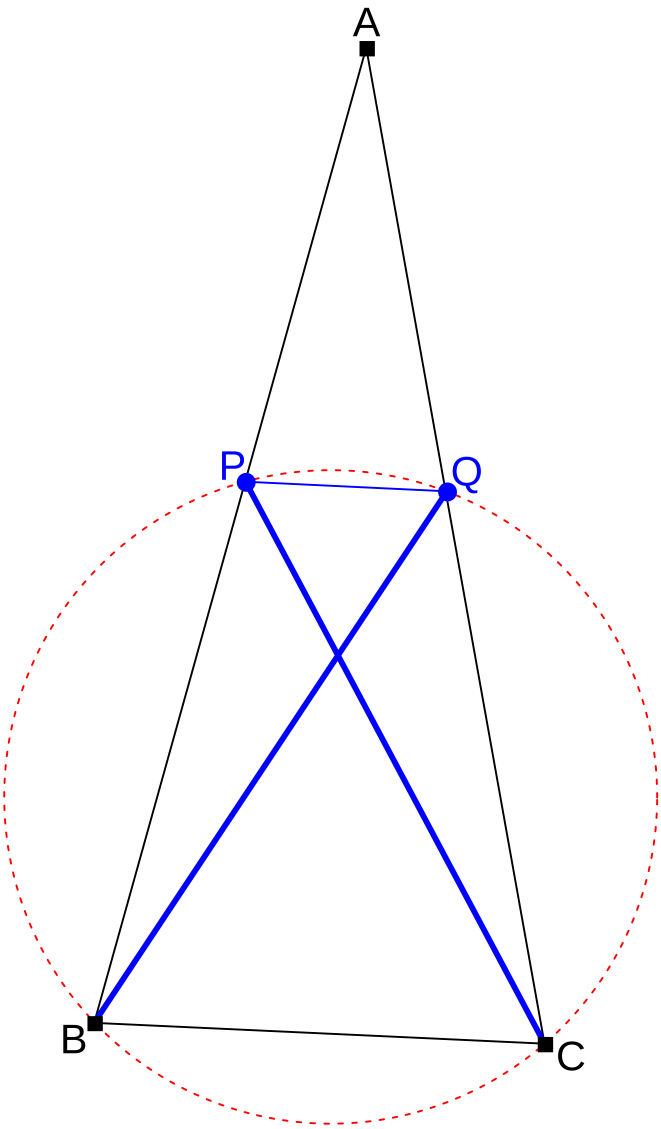


Fig. 6

(See Fig. 6.) The point P lies on the side AB of triangle ABC and satisfies $AP = 1$. Let Q be the point on the side AC of triangle ABC satisfying $AQ = 1$. Since the triangle ABC is isosceles with $AB = AC$, from symmetry it then follows that $PQ \parallel BC$, $BP = CQ$ and $BQ = CP$. Since $PQ \parallel BC$, we have $\measuredangle QPB = \pi - \measuredangle ABC$. Since triangle ABC is isosceles with $AB = AC$, we have $\measuredangle ABC = \measuredangle ACB$. Thus $\measuredangle QPB = \pi - \measuredangle ACB = \pi - \measuredangle QCB$. Thus, the quadrilateral $BPQC$ is cyclic, so the Ptolemy theorem yields $CP \bullet BQ = BC \bullet PQ + BP \bullet CQ$. Since $BC = 1$, $BQ = CP$ and $BP = CQ$, this becomes $CP \bullet CP = 1 \bullet PQ + BP \bullet BP$, what simplifies to $CP^2 = PQ + BP^2$.

The triangle ABC is isosceles with the base BC; let $\varphi = \measuredangle ABC = \measuredangle ACB$ be its base angle. Then, the angle at its apex A is $\measuredangle CAB = \pi - 2\varphi$. Consequently, $2\varphi = \pi - \measuredangle CAB$.

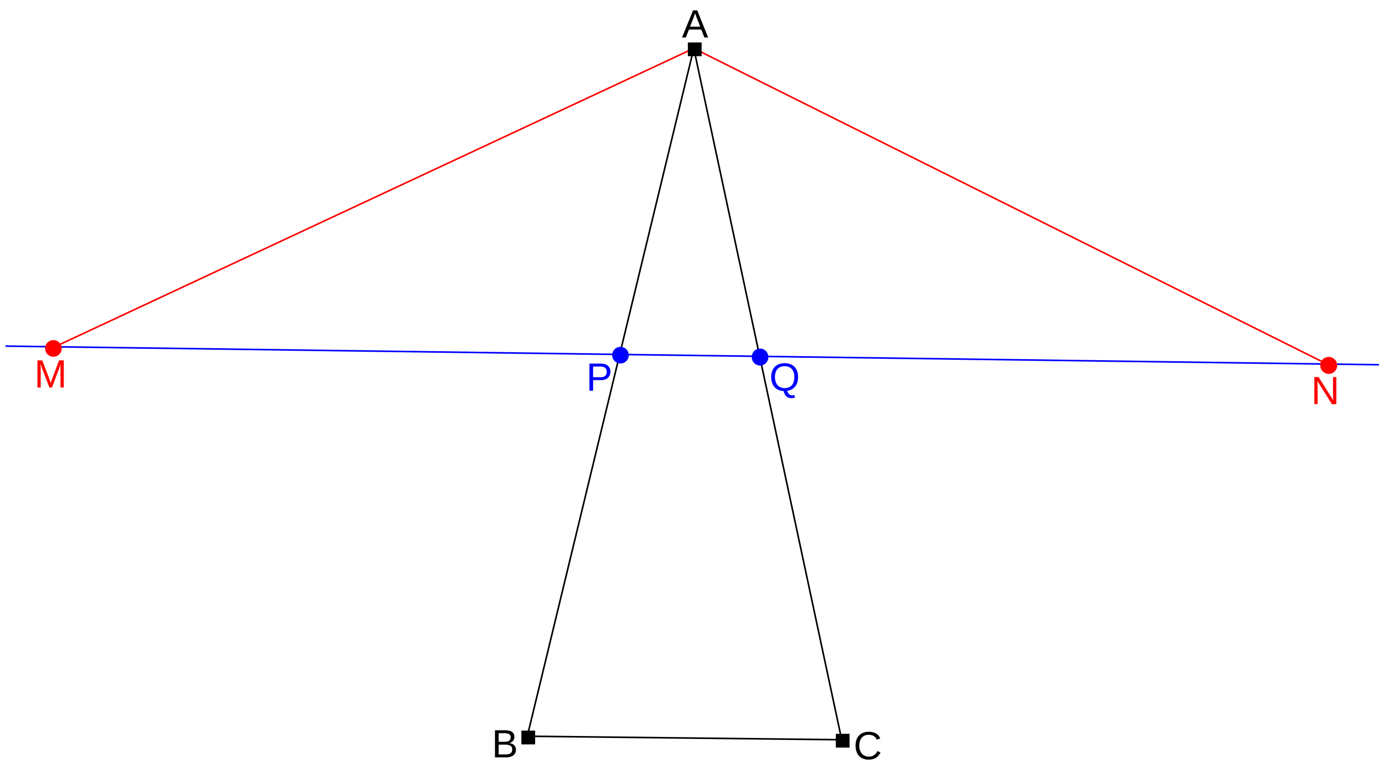
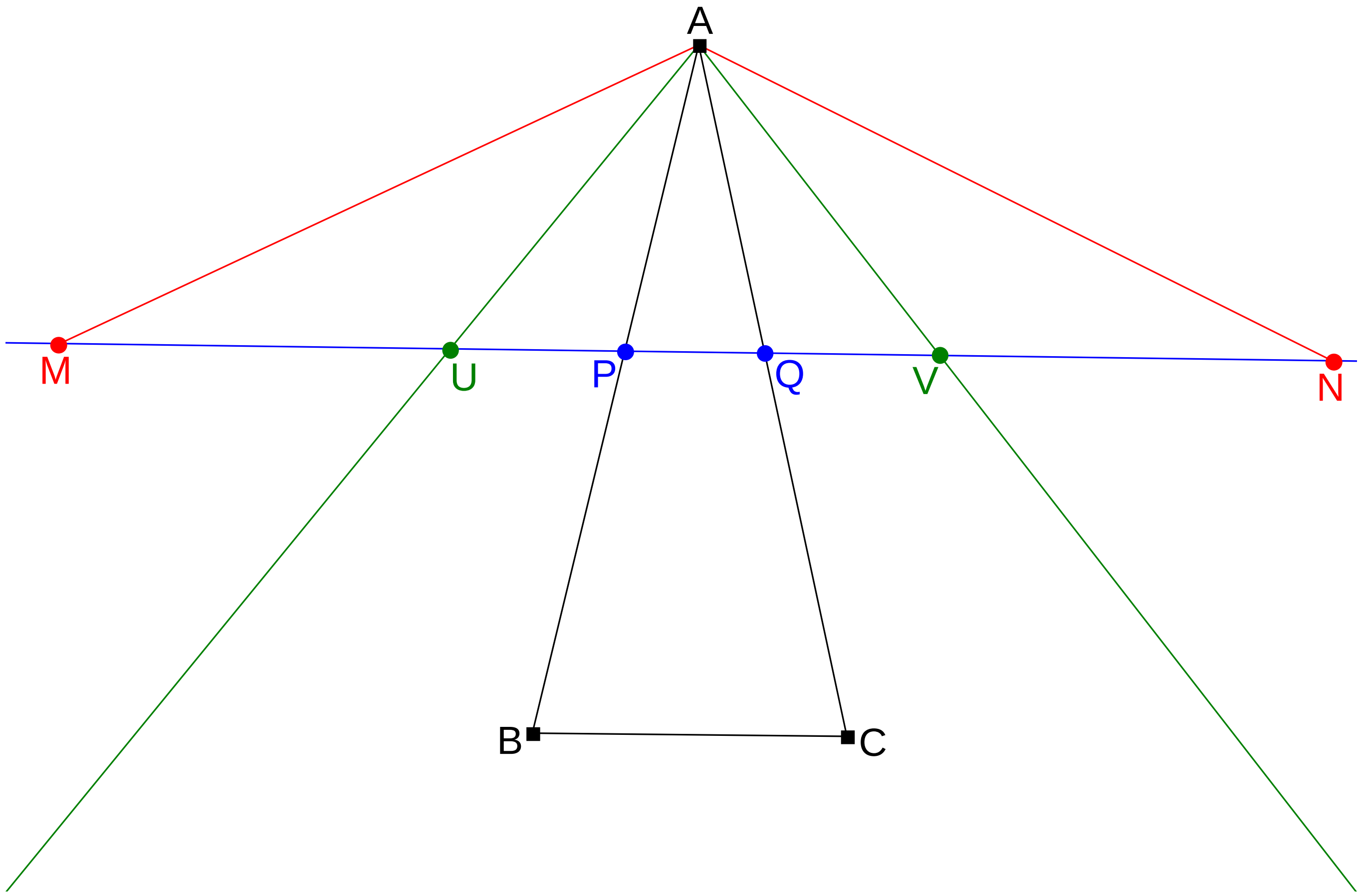


Fig. 7

(See Fig. 7.) Now let M be the point on the ray QP satisfying $\measuredangle MAQ = \varphi$. Since $PQ \parallel BC$, we have $\measuredangle AQM = \measuredangle ACB$, thus $\measuredangle AQM = \varphi$, and thus $\measuredangle MAQ = \measuredangle AQM = \varphi$; hence, the triangle MAQ is isosceles with base AQ, and it has the same base angle as the isosceles triangle ABC (in fact, the base angle of triangle ABC is φ, too). Furthermore, it has the same base as triangle ABC (since $AQ = 1$ and $BC = 1$). Hence, the isosceles triangle MAQ is congruent to the isosceles triangle ABC. Therefore, the legs of these two triangles are equal: $QM = AB$.

Since $\measuredangle CAB = \pi - 2\varphi$ and $\measuredangle MAQ = \varphi$, we have

$$\measuredangle MAP = \measuredangle MAQ - \measuredangle CAB = \varphi - (\pi - 2\varphi) = 3\varphi - \pi.$$



(See Fig. 8.) Let the angle bisector of the angle PAM intersect the line PQ at a point U. Then, $\measuredangle UAP = \frac{\measuredangle MAP}{2} = \frac{3\varphi - \pi}{2}$. Consequently,

$$\measuredangle UAQ = \measuredangle UAP + \measuredangle CAB = \frac{3\varphi - \pi}{2} + (\pi - 2\varphi) = \frac{(3\varphi - \pi) + 2 \bullet (\pi - 2\varphi)}{2} = \frac{\pi - \varphi}{2}.$$

On the other hand, $\measuredangle AQU = \measuredangle AQM = \varphi$; by the sum of angles in triangle UAQ, we thus have

$$\measuredangle AUQ = \pi - \measuredangle AQU - \measuredangle UAQ = \pi - \varphi - \frac{\pi - \varphi}{2} = \frac{\pi - \varphi}{2} = \measuredangle UAQ.$$

Therefore, the triangle UAQ is isosceles with $QU = AQ$. Since $AQ = 1$, this means that $QU = 1$. Together with $QM = AB$, this leads to $MU = QM - QU = AB - 1 = AB - AP = BP$.

Similarly to the point M on the ray QP satisfying $\measuredangle MAQ = \varphi$, we can construct a point N on the ray PQ satisfying $\measuredangle NAP = \varphi$. Similarly to the point U, we then define the point of intersection V of the angle bisector of the angle QAN with the line PQ. Similarly to the above equation $QU = 1$, we can now prove that $PV = 1$.

As showed above, $\measuredangle AUQ = \frac{\pi - \varphi}{2}$. In other words, $\measuredangle AUV = \frac{\pi - \varphi}{2}$. Similarly, $\measuredangle AVU = \frac{\pi - \varphi}{2}$. On the other hand, $\measuredangle UAQ = \frac{\pi - \varphi}{2}$ and $\measuredangle AUQ = \frac{\pi - \varphi}{2}$. Thus, $\measuredangle AUV = \measuredangle UAQ$ and $\measuredangle AVU = \measuredangle AUQ$. Hence, the triangles AUV and QAU are similar; this yields $AU : UV = QA : AU$, so that $AU^2 = QA \bullet UV$. Since $QA = AQ = 1$, this becomes $AU^2 = UV$.

Now, $UV = QU + QV = QU + (PV - PQ) = 1 + (1 - PQ) = 2 - PQ$, and hence

$$CP^2 - 2 = (PQ + BP^2) - 2 = BP^2 - (2 - PQ) = BP^2 - UV = MU^2 - AU^2$$

(since $MU = BP$ and $AU^2 = UV$).

As the triangle MAQ is congruent to the triangle ABC, we have $\measuredangle QMA = \measuredangle CAB$; in other words,

$\measuredangle UMA = \measuredangle CAB$. On the other hand, the line AU is the angle bisector of the angle PAM, and this yields

$$\measuredangle MAU = \frac{\measuredangle MAP}{2} = \frac{3\varphi - \pi}{2} = \frac{6\varphi - 2\pi}{4} = \frac{3 \bullet 2\varphi - 2\pi}{4}$$
$$= \frac{3 \bullet (\pi - \measuredangle CAB) - 2\pi}{4} = \frac{\pi - 3 \bullet \measuredangle CAB}{4}.$$

Now, we have $CP = \sqrt{2}$ if and only if $CP^2 = 2$. But since $CP^2 - 2 = MU^2 - AU^2$, we have $CP^2 = 2$ if and only if $MU^2 = AU^2$, thus if and only if $MU = AU$, i. e. if and only if the triangle AMU is isosceles with base AM. This, in turn, is equivalent to the equality of its angles $\measuredangle UMA$ and $\measuredangle MAU$; but because of $\measuredangle UMA = \measuredangle CAB$ and $\measuredangle MAU = \frac{\pi - 3 \bullet \measuredangle CAB}{4}$, these angles are equal if and only if $\measuredangle CAB = \frac{\pi - 3 \bullet \measuredangle CAB}{4}$. This simplifies to $4 \bullet \measuredangle CAB = \pi - 3 \bullet \measuredangle CAB$, and thus to $\measuredangle CAB = \frac{\pi}{7}$. Combining, we see that $CP = \sqrt{2}$ if and only if $\measuredangle CAB = \frac{\pi}{7}$; hence the problem is solved.