Problems from the Book – Problem 19.9

Let $n \in \mathbb{N}$. Let w_1, w_2, ..., w_n be n reals. Prove the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{ijw_iw_j}{i+j-1} \geq \left(\sum_{i=1}^{n} w_i \right)^2 .$$

Solution by Darij Grinberg

The following solution uses some linear algebra.
Notations.

- For any matrix A, we denote by $A\left[\begin{array}{c} j \\ i \end{array}\right]$ the entry in the j-th column and the i-th row of A. [This is usually denoted by A_{ij} or by $A_{i,j}$.]

- Let k be a field. Let $u \in \mathbb{N}$ and $v \in \mathbb{N}$, and let $a_{i,j}$ be an element of k for every $(i,j) \in \{1,2,...,u\} \times \{1,2,...,v\}$. Then, we denote by $(a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v}$ the $u \times v$ matrix A which satisfies $A\left[\begin{array}{c} j \\ i \end{array}\right] = a_{i,j}$ for every $(i,j) \in \{1,2,...,u\} \times \{1,2,...,v\}$.

- Let $n \in \mathbb{N}$. Let t_1, t_2, ..., t_n be n objects. Let $m \in \{1,2,...,n\}$. Then, we let $\left(t_1, t_2, ..., \widehat{t_m}, ..., t_n\right)$ denote the $(n-1)$-tuple $(t_1, t_2, ..., t_{m-2}, t_{m-1}, t_{m+1}, t_{m+2}, ..., t_n)$ (that is, the $(n-1)$-tuple $(s_1, s_2, ..., s_{n-1})$ defined by $s_i = \begin{cases} t_i, & \text{if } i < m; \\ t_{i+1}, & \text{if } i \geq m \end{cases}$ for all $i \in \{1,2,...,n-1\}$).

- Let L be a commutative ring with unity. Let T be a finite set. Let $a : T \to L$ be a map. Let $k \in \mathbb{N}$. We define an element $\sigma_k(a)$ of L by

  $$\sigma_k(a) = \sum_{\substack{S \subseteq T; \\ |S|=k}} \prod_{i \in S} a(i) .$$

  [Many readers will notice that if $T = \{1,2,...,n\}$ for some $n \in \mathbb{N}$, then $\sigma_k(a)$ is the k-th elementary symmetric polynomial evaluated at $a(1)$, $a(2)$, ..., $a(n)$.]
  The Viete theorem states that

  $$\prod_{\ell \in T} (x - a(\ell)) = \sum_{k=0}^{|T|} (-1)^k \sigma_k(a) x^{|T|-k} \qquad (1)$$

  for every $x \in L$.

  Theorem 1 (Sylvester). Let $n \in \mathbb{N}$, and let $A \in \mathbb{R}^{n \times n}$ be a symmetric $n \times n$ matrix. Then, the matrix A is positive definite if and only if every $m \in \{1,2,...,n\}$ satisfies $\det\left(\left(A\left[\begin{array}{c} j \\ i \end{array}\right]\right)_{1 \leq i \leq m}^{1 \leq j \leq m}\right) > 0$.

For a *proof of Theorem 1*, see any book on symmetric or Hermitian matrices.

Theorem 2 (Cauchy determinant). Let k be a field. Let $m \in \mathbb{N}$. Let $a_1, a_2, ..., a_m$ be m elements of k. Let $b_1, b_2, ..., b_m$ be m elements of k. Assume that $a_j \neq b_i$ for every $(i,j) \in \{1,2,...,m\}^2$. Then,

$$\det\left(\left(\frac{1}{a_j - b_i}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \frac{\prod\limits_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} \left((a_i - a_j)(b_j - b_i)\right)}{\prod\limits_{(i,j)\in\{1,2,...,m\}^2} (a_j - b_i)}.$$

In the following, I attempt to give the most conceptual proof of Theorem 2. First we recall a known fact we are not going to prove:

Theorem 3 (Vandermonde determinant). Let S be a commutative ring with unity. Let $m \in \mathbb{N}$. Let $a_1, a_2, ..., a_m$ be m elements of S. Then,

$$\det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j).$$

Besides, a trivial fact:

Lemma 4. Let S be a commutative ring with unity. Let $a \in S$. In the ring $S[X]$ (the polynomial ring over S in one indeterminate X), the element $X - a$ is not a zero divisor.

Proof of Lemma 4. Assume that $X - a$ is a zero divisor in $S[X]$. Then, there exists a polynomial $P(X) \in S[X]$ such that $(X-a)P(X) = 0$ and $P(X) \neq 0$. Let $n = \deg P$; then, there exist $n+1$ elements $r_0, r_1, ..., r_n$ of S such that $r_n \neq 0$ and $P(X) = \sum_{k=0}^n r_k X^k$. Define $r_{n+1} \in S$ by $r_{n+1} = 0$. Define $r_{-1} \in S$ by $r_{-1} = 0$. Then,

$$\sum_{k=0}^{n+1} r_k X^k = \sum_{k=0}^{n} r_k X^k + \underbrace{r_{n+1}}_{=0} X^{n+1} = \sum_{k=0}^{n} r_k X^k + 0 = \sum_{k=0}^{n} r_k X^k = P(X)$$

and

$$\begin{aligned}\sum_{k=0}^{n+1} r_{k-1} X^k &= \underbrace{r_{0-1}}_{=r_{-1}=0} X^0 + \sum_{k=1}^{n+1} r_{k-1} X^k = 0 + \sum_{k=1}^{n+1} r_{k-1} X^k = \sum_{k=1}^{n+1} r_{k-1} X^k \\ &= \sum_{k=0}^{n} \underbrace{r_{(k+1)-1}}_{=r_k} \underbrace{X^{k+1}}_{=X^k X} \qquad \text{(here we substituted } k+1 \text{ for } k \text{ in the sum)} \\ &= X \sum_{k=0}^{n} r_k X^k = XP(X).\end{aligned}$$

Hence,

$$0 = (X-a)\,P(X) = \underbrace{XP(X)}_{=\sum_{k=0}^{n+1} r_{k-1}X^k} - a \underbrace{P(X)}_{=\sum_{k=0}^{n+1} r_k X^k}$$
$$= \sum_{k=0}^{n+1} r_{k-1}X^k - a\sum_{k=0}^{n+1} r_k X^k = \sum_{k=0}^{n+1} (r_{k-1} - ar_k)\, X^k.$$

Since $r_{k-1} - ar_k \in S$ for every $k \in \{0, 1, ..., n+1\}$, this yields $r_{k-1} - ar_k = 0$ for every $k \in \{0, 1, ..., n+1\}$. For $k = n+1$, this yields $r_{(n+1)-1} - ar_{n+1} = 0$. Thus,

$$0 = \underbrace{r_{(n+1)-1}}_{=r_n} - a \underbrace{r_{n+1}}_{=0} = r_n - a \cdot 0 = r_n,$$

what contradicts $r_n \neq 0$. Hence, our assumption that $X - a$ is a zero divisor in $S[X]$ was wrong. Therefore, $X - a$ is not a zero divisor in $S[X]$. This proves Lemma 4.

Lemma 5. Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. In the ring $R[X_1, X_2, ..., X_m]$ (the polynomial ring over R in m indeterminates X_1, X_2, ..., X_m), the element $\prod\limits_{\substack{(i,j) \in \{1,2,...,m\}^2;\\ i>j}} (X_i - X_j)$ is not a zero divisor.

Proof of Lemma 5. We will first show that:

For any $(i,j) \in \{1, 2, ..., m\}^2$ satisfying $i > j$, the element $X_i - X_j$ of the ring $R[X_1, X_2, ..., X_m]$ is not a zero divisor. (2)

Proof of (2). Let $R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right]$ denote the sub-R-algebra of $R[X_1, X_2, ..., X_m]$ generated by the $m-1$ elements X_1, X_2, ..., X_{i-2}, X_{i-1}, X_{i+1}, X_{i+2}, ..., X_m (that is, the m elements X_1, X_2, ..., X_m except of X_i). (In other words, define a sub-R-algebra $R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right]$ of $R[X_1, X_2, ..., X_m]$ by $R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right] = R[y_1, y_2, ..., y_{m-1}]$, where we define $m-1$ elements y_1, y_2, ..., y_{m-1} of $R[X_1, X_2, ..., X_m]$ by $y_j = \begin{cases} X_j, & \text{if } j < i; \\ X_{j+1}, & \text{if } j \geq i \end{cases}$ for every $j \in \{1, 2, ..., m-1\}$.)

Consider the ring $\left(R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right]\right)[X]$ (this is the polynomial ring over the ring $R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right]$ in one indeterminate X).

It is known that there exists an R-algebra isomorphism $\phi : \left(R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right]\right)[X] \to R[X_1, X_2, ..., X_m]$ such that $\phi(X) = X_i$ and $\phi(X_k) = X_k$ for every $k \in \{1, 2, ..., m\} \setminus \{i\}$.

Since $i > j$ yields $j \in \{1, 2, ..., m\} \setminus \{i\}$, we have $\phi(X_j) = X_j$ and thus $\phi(X - X_j) = \underbrace{\phi(X)}_{=X_i} - \underbrace{\phi(X_j)}_{=X_j} = X_i - X_j$. Since $X - X_j$ is not a zero divisor in $\left(R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right]\right)[X]$ (by Lemma 4, applied to $S = R\left[X_1, X_2, ..., \widehat{X_i}, ..., X_m\right]$ and $a = X_j$), it follows that $\phi(X - X_j)$ is not a zero divisor in $R[X_1, X_2, ..., X_m]$ (since ϕ is an R-algebra isomorphism). In other words, $X_i - X_j$ is not a zero divisor in $R[X_1, X_2, ..., X_m]$ (since $\phi(X - X_j) = X_i - X_j$). This proves (2).

It is known that if we choose some elements of a ring such that each of these elements is not a zero divisor, then the product of these elements is not a zero divisor. Hence, (2) yields that the element $\prod\limits_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j)$ of the ring $R[X_1, X_2, ..., X_m]$ is not a zero divisor. This proves Lemma 5.

Now comes a rather useful fact:

Theorem 6. Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. Consider the ring $R[X_1, X_2, ..., X_m]$ (the polynomial ring over R in m indeterminates X_1, X_2, ..., X_m). Define a map $X : \{1, 2, ..., m\} \to R[X_1, X_2, ..., X_m]$ by $X(i) = X_i$ for every $i \in \{1, 2, ..., m\}$. Then,

$$\det\left(\left((-1)^{m-j}\sigma_{m-j}\left(X\mid_{\{1,2,...,m\}\setminus\{i\}}\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}} (X_i - X_j).$$

Proof of Theorem 6. Theorem 3 (applied to $S = R[X_1, X_2, ..., X_m]$ and $a_i = X_i$) yields

$$\det\left(\left(X_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j). \tag{3}$$

Let $V = \left(X_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}$. Then, $V\begin{bmatrix} j \\ i \end{bmatrix} = X_i^{j-1}$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$.

Since $\left(X_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m} = V$, the equation (3) becomes

$$\det V = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j).$$

Let $W = \left((-1)^{m-j}\sigma_{m-j}\left(X\mid_{\{1,2,...,m\}\setminus\{i\}}\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}$. Then, $W\begin{bmatrix} j \\ i \end{bmatrix} = (-1)^{m-j}\sigma_{m-j}\left(X\mid_{\{1,2,...,m\}\setminus\{i\}}\right)$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$.

For every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, we can apply (1) to $L = R[X_1, X_2, ..., X_m]$, $x = X_j$, $T = \{1, 2, ..., m\} \setminus \{i\}$ and $a = X\mid_{\{1,2,...,m\}\setminus\{i\}}$, and obtain

$$\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} \left(X_j - \left(X\mid_{\{1,2,...,m\}\setminus\{i\}}\right)(\ell)\right) = \sum_{k=0}^{m-1} (-1)^k \sigma_k\left(X\mid_{\{1,2,...,m\}\setminus\{i\}}\right) X_j^{(m-1)-k},$$

because

$$\begin{aligned} |T| &= |\{1, 2, ..., m\} \setminus \{i\}| = |\{1, 2, ..., m\}| - 1 \qquad (\text{since } i \in \{1, 2, ..., m\}) \\ &= m - 1. \end{aligned}$$

Since $\left(X\mid_{\{1,2,...,m\}\setminus\{i\}}\right)(\ell) = X(\ell) = X_\ell$ and $X_j^{(m-1)-k} = X_j^{(m-k)-1}$, this becomes

$$\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (X_j - X_\ell) = \sum_{k=0}^{m-1} (-1)^k \sigma_k\left(X\mid_{\{1,2,...,m\}\setminus\{i\}}\right) X_j^{(m-k)-1}. \tag{4}$$

Now, for every $i \in \{1,2,...,m\}$ and $j \in \{1,2,...,m\}$, we have

$$\left(WV^T\right)\begin{bmatrix} j \\ i \end{bmatrix} = \sum_{k=1}^{m} \underbrace{W\begin{bmatrix} k \\ i \end{bmatrix}}_{=(-1)^{m-k}\sigma_{m-k}\left(X|_{\{1,2,...,m\}\setminus\{i\}}\right)} \cdot \underbrace{V^T\begin{bmatrix} j \\ k \end{bmatrix}}_{=V\begin{bmatrix} k \\ j \end{bmatrix}=X_j^{k-1}}$$

$$= \sum_{k=1}^{m} (-1)^{m-k}\,\sigma_{m-k}\left(X\mid_{\{1,2,...,m\}\setminus\{i\}}\right) X_j^{k-1}$$

$$= \sum_{k=0}^{m-1} (-1)^{k}\,\sigma_{k}\left(X\mid_{\{1,2,...,m\}\setminus\{i\}}\right) X_j^{(m-k)-1}$$

(here, we substituted k for $m-k$ in the sum)

$$= \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} \left(X_j - X_\ell\right) \qquad \text{(by (4))}. \qquad (5)$$

Thus, if $j \neq i$, then

$$\left(WV^T\right)\begin{bmatrix} j \\ i \end{bmatrix} = \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} \left(X_j - X_\ell\right) = \underbrace{\left(X_j - X_j\right)}_{=0} \cdot \prod_{\ell\in(\{1,2,...,m\}\setminus\{i\})\setminus\{j\}} \left(X_j - X_\ell\right)$$

(since $j \in \{1,2,...,m\}\setminus\{i\}$, because $j \in \{1,2,...,m\}$ and $j \neq i$)

$$= 0.$$

Hence, the matrix WV^T is diagonal. Therefore,

$$\det\left(WV^T\right) = \prod_{i=1}^{m} \left(WV^T\right)\begin{bmatrix} i \\ i \end{bmatrix} = \prod_{i=1}^{m} \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} \left(X_i - X_\ell\right)$$

$$\left(\text{since (5), applied to } j = i\text{, yields } \left(WV^T\right)\begin{bmatrix} i \\ i \end{bmatrix} = \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} \left(X_i - X_\ell\right)\right)$$

$$= \prod_{i\in\{1,2,...,m\}} \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} \left(X_i - X_\ell\right)$$

$$= \prod_{i\in\{1,2,...,m\}} \prod_{j\in\{1,2,...,m\}\setminus\{i\}} \left(X_i - X_j\right) \qquad \text{(here, we renamed } \ell \text{ as } j \text{ in the second product)}$$

$$= \prod_{i\in\{1,2,...,m\}} \prod_{\substack{j\in\{1,2,...,m\};\\ j\neq i}} \left(X_i - X_j\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j\neq i}} \left(X_i - X_j\right)$$

$$= \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}} \left(X_i - X_j\right) \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} \left(X_i - X_j\right)$$

$$\left(\begin{array}{c}\text{since the set } \left\{(i,j)\in\{1,2,...,m\}^2 \mid j\neq i\right\} \text{ is the union of the two disjoint sets} \\ \left\{(i,j)\in\{1,2,...,m\}^2 \mid j>i\right\} \text{ and } \left\{(i,j)\in\{1,2,...,m\}^2 \mid i>j\right\}\end{array}\right).$$

But on the other hand,

$$\det\left(WV^T\right) = \det W \cdot \det\left(V^T\right)$$
$$= \det W \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j) \qquad \left(\text{since } \det\left(V^T\right) = \det V = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j)\right).$$

Hence,

$$\det W \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j) = \det\left(WV^T\right)$$
$$= \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}} (X_i - X_j) \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j).$$

But since the element $\prod\limits_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (X_i - X_j)$ of the ring $R[X_1, X_2, ..., X_m]$ is not a zero divisor (according to Lemma 5), this yields

$$\det W = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}} (X_i - X_j).$$

Since $W = \left((-1)^{m-j} \sigma_{m-j}\left(X\mid_{\{1,2,...,m\}\setminus\{i\}}\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}$, this becomes

$$\det\left(\left((-1)^{m-j} \sigma_{m-j}\left(X\mid_{\{1,2,...,m\}\setminus\{i\}}\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}} (X_i - X_j).$$

Thus, Theorem 6 is proven.

Next, we show:

Theorem 7. Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. Let a_1, a_2, ..., a_m be m elements of R. Let b_1, b_2, ..., b_m be m elements of R. Then,

$$\det\left(\left(\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - b_\ell)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} ((a_i - a_j)(b_j - b_i)).$$

Proof of Theorem 7. Consider the ring $R[X_1, X_2, ..., X_m]$ (the polynomial ring over R in m indeterminates X_1, X_2, ..., X_m). Define a map $X : \{1,2,...,m\} \to R[X_1, X_2, ..., X_m]$ by $X(i) = X_i$ for every $i \in \{1,2,...,m\}$.

Let $\widetilde{V} = \left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}$. Then, $\widetilde{V}\begin{bmatrix} j \\ i \end{bmatrix} = a_i^{j-1}$ for every $i \in \{1,2,...,m\}$ and $j \in \{1,2,...,m\}$. Besides,

$$\det \widetilde{V} = \det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j) \qquad (\text{by Theorem 3}).$$

Let $W = \left((-1)^{m-j} \sigma_{m-j} \left(X \mid_{\{1,2,...,m\} \setminus \{i\}} \right) \right)_{1 \leq i \leq m}^{1 \leq j \leq m}$. Then, $W \left[\begin{array}{c} j \\ i \end{array} \right] = (-1)^{m-j} \sigma_{m-j} \left(X \mid_{\{1,2,...,m\} \setminus \{i\}} \right)$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$.

For every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, we can apply (1) to $L = R[X_1, X_2, ..., X_m]$, $x = a_j$, $T = \{1, 2, ..., m\} \setminus \{i\}$ and $a = X \mid_{\{1,2,...,m\} \setminus \{i\}}$, and obtain

$$\prod_{\ell \in \{1,2,...,m\} \setminus \{i\}} \left(a_j - \left(X \mid_{\{1,2,...,m\} \setminus \{i\}} \right) (\ell) \right) = \sum_{k=0}^{m-1} (-1)^k \sigma_k \left(X \mid_{\{1,2,...,m\} \setminus \{i\}} \right) a_j^{(m-1)-k},$$

because

$$|T| = |\{1, 2, ..., m\} \setminus \{i\}| = |\{1, 2, ..., m\}| - 1 \qquad (\text{since } i \in \{1, 2, ..., m\})$$
$$= m - 1.$$

Since $\left(X \mid_{\{1,2,...,m\} \setminus \{i\}} \right) (\ell) = X(\ell) = X_\ell$ and $a_j^{(m-1)-k} = a_j^{(m-k)-1}$, this becomes

$$\prod_{\ell \in \{1,2,...,m\} \setminus \{i\}} (a_j - X_\ell) = \sum_{k=0}^{m-1} (-1)^k \sigma_k \left(X \mid_{\{1,2,...,m\} \setminus \{i\}} \right) a_j^{(m-k)-1}. \qquad (6)$$

Now, for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, we have

$$\left(W \widetilde{V}^T \right) \left[\begin{array}{c} j \\ i \end{array} \right] = \sum_{k=1}^{m} \underbrace{W \left[\begin{array}{c} k \\ i \end{array} \right]}_{= (-1)^{m-k} \sigma_{m-k} \left(X \mid_{\{1,2,...,m\} \setminus \{i\}} \right)} \cdot \underbrace{\widetilde{V}^T \left[\begin{array}{c} j \\ k \end{array} \right]}_{= \widetilde{V} \left[\begin{array}{c} k \\ j \end{array} \right] = a_j^{k-1}}$$
$$= \sum_{k=1}^{m} (-1)^{m-k} \sigma_{m-k} \left(X \mid_{\{1,2,...,m\} \setminus \{i\}} \right) a_j^{k-1}$$
$$= \sum_{k=0}^{m-1} (-1)^k \sigma_k \left(X \mid_{\{1,2,...,m\} \setminus \{i\}} \right) a_j^{(m-k)-1}$$
$$(\text{here, we substituted } k \text{ for } m - k \text{ in the sum})$$
$$= \prod_{\ell \in \{1,2,...,m\} \setminus \{i\}} (a_j - X_\ell) \qquad (\text{by (6)}).$$

Hence,

$$W \widetilde{V}^T = \left(\prod_{\ell \in \{1,2,...,m\} \setminus \{i\}} (a_j - X_\ell) \right)_{1 \leq i \leq m}^{1 \leq j \leq m}.$$

Thus,

$$\det\left(\underbrace{\left(\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}}(a_j-X_\ell)\right)_{1\leq i\leq m}^{1\leq j\leq m}}_{=W\widetilde{V}^T}\right)=\det\left(W\widetilde{V}^T\right)=\det W\cdot\det\left(\widetilde{V}^T\right)$$

$$=\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}}(X_i-X_j)\cdot\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}(a_i-a_j)$$

$$\left(\begin{array}{c}\text{since }\det W=\det\left(\left((-1)^{m-j}\sigma_{m-j}\left(X\mid_{\{1,2,...,m\}\setminus\{i\}}\right)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)=\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j>i}}(X_i-X_j)\\ \text{by Theorem 6 and }\det\left(\widetilde{V}^T\right)=\det\widetilde{V}=\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}(a_i-a_j)\end{array}\right)$$

$$=\prod_{\substack{(j,i)\in\{1,2,...,m\}^2;\\ i>j}}(X_j-X_i)\cdot\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}(a_i-a_j)$$

(here, we renamed i and j as j and i in the first product)

$$=\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}(X_j-X_i)\cdot\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}(a_i-a_j)$$

$\left(\text{since } (j,i)\in\{1,2,...,m\}^2 \text{ is equivalent to } (i,j)\in\{1,2,...,m\}^2\right)$

$$=\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}((X_j-X_i)(a_i-a_j))=\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}((a_i-a_j)(X_j-X_i)).$$

Both sides of this identity are polynomials over the ring R in m indeterminates X_1, X_2, ..., X_m. Evaluating these polynomials at $X_1=b_1$, $X_2=b_2$, ..., $X_m=b_m$, we obtain

$$\det\left(\left(\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}}(a_j-b_\ell)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)=\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}((a_i-a_j)(b_j-b_i)).$$

This proves Theorem 7.

Proof of Theorem 2. Let $Q=\left(\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}}(a_j-b_\ell)\right)_{1\leq i\leq m}^{1\leq j\leq m}$. Then, $Q\left[\begin{array}{c}j\\ i\end{array}\right]=\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}}(a_j-b_\ell)$ for every $i\in\{1,2,...,m\}$ and $j\in\{1,2,...,m\}$. Also,

$$\det Q=\det\left(\left(\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}}(a_j-b_\ell)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)=\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}((a_i-a_j)(b_j-b_i))\qquad\text{(by Theorem 7)}.$$

Let $P=\left(\left\{\begin{array}{c}\prod_{\ell\in\{1,2,...,m\}}(a_i-b_\ell),\text{ if } i=j;\\ 0,\text{ if } i\neq j\end{array}\right.\right)_{1\leq i\leq m}^{1\leq j\leq m}$. Then,

$P\begin{bmatrix} j \\ i \end{bmatrix} = \begin{cases} \left(\prod_{\ell\in\{1,2,...,m\}} (a_i - b_\ell)\right)^{-1}, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases}$ for every $i \in \{1,2,...,m\}$ and $j \in \{1,2,...,m\}$. Thus, P is a diagonal matrix, so that

$$\det P = \prod_{j=1}^{m} P\begin{bmatrix} j \\ j \end{bmatrix} = \prod_{j\in\{1,2,...,m\}} \underbrace{P\begin{bmatrix} j \\ j \end{bmatrix}}_{\substack{=\left(\prod_{\ell\in\{1,2,...,m\}}(a_j - b_\ell)\right)^{-1}, \\ \text{since } j=j}}$$

$$= \prod_{j\in\{1,2,...,m\}} \left(\prod_{\ell\in\{1,2,...,m\}} (a_j - b_\ell)\right)^{-1} = \left(\prod_{j\in\{1,2,...,m\}} \prod_{\ell\in\{1,2,...,m\}} (a_j - b_\ell)\right)^{-1}$$

$$= \left(\prod_{(\ell,j)\in\{1,2,...,m\}^2} (a_j - b_\ell)\right)^{-1} = \left(\prod_{(i,j)\in\{1,2,...,m\}^2} (a_j - b_i)\right)^{-1}$$

$$(\text{here, we renamed } \ell \text{ as } i \text{ in the product}).$$

Now, for every $i \in \{1,2,...,m\}$ and $j \in \{1,2,...,m\}$, we have

$$(QP)\begin{bmatrix} j \\ i \end{bmatrix} = \sum_{k=1}^{m} Q\begin{bmatrix} k \\ i \end{bmatrix} P\begin{bmatrix} j \\ k \end{bmatrix} = \sum_{k\in\{1,2,...,m\}} Q\begin{bmatrix} k \\ i \end{bmatrix} P\begin{bmatrix} j \\ k \end{bmatrix}$$

$$= \sum_{k\in\{1,2,...,m\}\setminus\{j\}} Q\begin{bmatrix} k \\ i \end{bmatrix} \underbrace{P\begin{bmatrix} j \\ k \end{bmatrix}}_{\substack{=0, \text{ since} \\ k\in\{1,2,...,m\}\setminus\{j\} \\ \text{yields } k\neq j}} + \underbrace{\sum_{k\in\{j\}} Q\begin{bmatrix} k \\ i \end{bmatrix} P\begin{bmatrix} j \\ k \end{bmatrix}}_{=Q\begin{bmatrix} j \\ i \end{bmatrix} P\begin{bmatrix} j \\ j \end{bmatrix}}$$

$$= \underbrace{\sum_{k\in\{1,2,...,m\}\setminus\{j\}} Q\begin{bmatrix} k \\ i \end{bmatrix} \cdot 0}_{=0} + Q\begin{bmatrix} j \\ i \end{bmatrix} P\begin{bmatrix} j \\ j \end{bmatrix}$$

$$= \underbrace{Q\begin{bmatrix} j \\ i \end{bmatrix}}_{=\prod_{\ell\in\{1,2,...,m\}\setminus\{i\}}(a_j - b_\ell)} \underbrace{P\begin{bmatrix} j \\ j \end{bmatrix}}_{\substack{=\left(\prod_{\ell\in\{1,2,...,m\}}(a_j - b_\ell)\right)^{-1}, \\ \text{since } j=j}}$$

$$= \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - b_\ell) \cdot \left(\prod_{\ell\in\{1,2,...,m\}} (a_j - b_\ell)\right)^{-1}$$

$$= \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - b_\ell) \cdot \left((a_j - b_i) \cdot \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - b_\ell)\right)^{-1}$$

$$\left(\text{since } \prod_{\ell\in\{1,2,...,m\}} (a_j - b_\ell) = (a_j - b_i) \cdot \prod_{\ell\in\{1,2,...,m\}\setminus\{i\}} (a_j - b_\ell)\right)$$

$$= (a_j - b_i)^{-1} = \frac{1}{a_j - b_i}.$$

Thus,

$$QP = \left(\frac{1}{a_j - b_i}\right)_{1\leq i\leq m}^{1\leq j\leq m}.$$

Hence,

$$\det\left(\underbrace{\left(\frac{1}{a_j - b_i}\right)_{1\leq i\leq m}^{1\leq j\leq m}}_{=QP}\right) = \det(QP) = \det Q \cdot \det P$$

$$= \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} \left((a_i - a_j)(b_j - b_i)\right) \cdot \left(\prod_{(i,j)\in\{1,2,...,m\}^2} (a_j - b_i)\right)^{-1}$$

$$\left(\text{since } \det Q = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} \left((a_i - a_j)(b_j - b_i)\right) \text{ and } \det P = \left(\prod_{(i,j)\in\{1,2,...,m\}^2} (a_j - b_i)\right)^{-1}\right)$$

$$= \frac{\prod\limits_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} \left((a_i - a_j)(b_j - b_i)\right)}{\prod_{(i,j)\in\{1,2,...,m\}^2} (a_j - b_i)}.$$

Thus, Theorem 2 is proven.

Theorem 8. Let $n \in \mathbb{N}$. Let a_1, a_2, ..., a_n be n pairwise distinct reals. Let c be a real such that $a_i + a_j + c > 0$ for every $(i, j) \in \{1, 2, ..., n\}^2$. Then, the matrix $\left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n} \in \mathbb{R}^{n\times n}$ is positive definite.

Proof of Theorem 8. Let $A = \left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n}$. Then, $A\begin{bmatrix} j \\ i \end{bmatrix} = \frac{1}{a_i + a_j + c}$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$.

Thus, $A \in \mathbb{R}^{n\times n}$ is a symmetric $n \times n$ matrix (since $A\begin{bmatrix} j \\ i \end{bmatrix} = \frac{1}{a_i + a_j + c} = \frac{1}{a_j + a_i + c} = A\begin{bmatrix} i \\ j \end{bmatrix}$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$).

Define n reals b_1, b_2, ..., b_n by $b_i = -a_i - c$ for every $i \in \{1, 2, ..., n\}$.

Let $m \in \{1, 2, ..., n\}$. Then, $a_j \neq b_i$ for every $(i, j) \in \{1, 2, ..., m\}^2$ (since $a_j - b_i = a_j - (-a_i - c) = a_i + a_j + c > 0$ yields $a_j > b_i$). Thus, Theorem 2 (applied to $k = \mathbb{R}$) yields

$$\det\left(\left(\frac{1}{a_j - b_i}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \frac{\prod\limits_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} \left((a_i - a_j)(b_j - b_i)\right)}{\prod\limits_{(i,j)\in\{1,2,...,m\}^2} (a_j - b_i)}.$$

Thus, every $m \in \{1, 2, ..., n\}$ satisfies

$$\det\left(\left(A\left[\begin{array}{c} j \\ i \end{array}\right]\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \det\left(\left(\frac{1}{a_j - b_i}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)$$

$$\left(\text{since } A\left[\begin{array}{c} j \\ i \end{array}\right] = \frac{1}{a_i + a_j + c} = \frac{1}{a_j - (-a_i - c)} = \frac{1}{a_j - b_i}\right)$$

$$= \frac{\prod\limits_{\substack{(i,j)\in\{1,2,...,m\}^2; \\ i>j}} \left((a_i - a_j)(b_j - b_i)\right)}{\prod\limits_{(i,j)\in\{1,2,...,m\}^2} (a_j - b_i)} = \frac{\prod\limits_{\substack{(i,j)\in\{1,2,...,m\}^2; \\ i>j}} (a_i - a_j)^2}{\prod\limits_{(i,j)\in\{1,2,...,m\}^2} (a_i + a_j + c)}$$

$$\left(\begin{array}{c} \text{since } (a_i - a_j)(b_j - b_i) = (a_i - a_j)\underbrace{((-a_j - c) - (-a_i - c))}_{=-a_j-c+a_i+c=a_i-a_j} = (a_i - a_j)^2 \\ \text{and } a_j - b_i = a_j - (-a_i - c) = a_i + a_j + c \end{array}\right)$$

$$> 0$$

(since $(a_i - a_j)^2 > 0$ for every $(i, j) \in \{1, 2, ..., m\}^2$ satisfying $i > j$ (because a_1, a_2, ..., a_n are pairwise distinct, so that $a_i \neq a_j$, thus $a_i - a_j \neq 0$ and therefore $(a_i - a_j)^2 > 0$), and $a_i + a_j + c > 0$ for every $(i, j) \in \{1, 2, ..., m\}^2$).

Hence, according to Theorem 1, the symmetric matrix A is positive definite. Since $A = \left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n}$, this means that the matrix $\left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n}$ is positive definite. Thus, Theorem 8 is proven.

Corollary 9. Let $n \in \mathbb{N}$. Let a_1, a_2, ..., a_n be n pairwise distinct reals. Let c be a real such that $a_i + a_j + c > 0$ for every $(i, j) \in \{1, 2, ..., n\}^2$. Let v_1, v_2, ..., v_n be n reals. Then, the inequality $\sum_{i=1}^n \sum_{j=1}^n \frac{v_i v_j}{a_i + a_j + c} \geq 0$ holds, with equality if and only if $v_1 = v_2 = ... = v_n = 0$.

Proof of Corollary 9. Define a vector $v \in \mathbb{R}^n$ by $v = \left(\begin{array}{c} v_1 \\ v_2 \\ ... \\ v_n \end{array}\right)$. Then,

$$v^T \left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n} v = \sum_{i=1}^n \sum_{j=1}^n \frac{1}{a_i + a_j + c} v_i v_j = \sum_{i=1}^n \sum_{j=1}^n \frac{v_i v_j}{a_i + a_j + c}. \qquad (7)$$

Also, obviously,

$$v = 0 \text{ holds if and only if } v_1 = v_2 = ... = v_n = 0. \qquad (8)$$

Now, since the matrix $\left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n} \in \mathbb{R}^{n\times n}$ is positive definite (by Theorem 8), we have $v^T \left(\frac{1}{a_i + a_j + c}\right)_{1\leq i\leq n}^{1\leq j\leq n} v \geq 0$, with equality if and only if $v = 0$. According to (7) and (8), this means that $\sum_{i=1}^n \sum_{j=1}^n \frac{v_i v_j}{a_i + a_j + c} \geq 0$, with equality if and only if $v_1 = v_2 = ... = v_n = 0$. Thus, Corollary 9 is proven.

Corollary 10. Let $n \in \mathbb{N}$. Let $a_1, a_2, ..., a_n$ be n pairwise distinct reals. Let c be a real such that $a_i + a_j + c > 0$ for every $(i, j) \in \{1, 2, ..., n\}^2$. Let $w_1, w_2, ..., w_n$ be n reals. Then, the inequality $\sum_{i=1}^n \sum_{j=1}^n \frac{a_i a_j w_i w_j}{a_i + a_j + c} \geq -c \left(\sum_{i=1}^n w_i\right)^2$ holds, with equality if and only if $(c + a_1) w_1 = (c + a_2) w_2 = ... = (c + a_n) w_n = 0$.

Proof of Corollary 10. Define n reals $v_1, v_2, ..., v_n$ by $v_i = (c + a_i) w_i$ for every $i \in \{1, 2, ..., n\}$.

Then,

$$\sum_{i=1}^n \sum_{j=1}^n \frac{a_i a_j w_i w_j}{a_i + a_j + c} - \left(-c \left(\sum_{i=1}^n w_i\right)^2\right)$$

$$= \sum_{i=1}^n \sum_{j=1}^n \frac{a_i a_j w_i w_j}{a_i + a_j + c} + c \left(\sum_{i=1}^n w_i\right)^2 = \sum_{i=1}^n \sum_{j=1}^n \frac{a_i a_j w_i w_j}{a_i + a_j + c} + c \sum_{i=1}^n \sum_{j=1}^n w_i w_j$$

$$\left(\text{since } \left(\sum_{i=1}^n w_i\right)^2 = \sum_{i=1}^n w_i \cdot \sum_{i=1}^n w_i = \sum_{i=1}^n w_i \cdot \sum_{j=1}^n w_j = \sum_{i=1}^n \sum_{j=1}^n w_i w_j\right)$$

$$= \sum_{i=1}^n \sum_{j=1}^n \left(\frac{a_i a_j w_i w_j}{a_i + a_j + c} + c w_i w_j\right) = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{a_i a_j}{a_i + a_j + c} + c\right) w_i w_j$$

$$= \sum_{i=1}^n \sum_{j=1}^n \frac{a_i a_j + (a_i + a_j + c) c}{a_i + a_j + c} w_i w_j = \sum_{i=1}^n \sum_{j=1}^n \frac{(c + a_i)(c + a_j)}{a_i + a_j + c} w_i w_j$$

$$\left(\text{since } a_i a_j + (a_i + a_j + c) c = a_i a_j + a_i c + a_j c + c^2 = (c + a_i)(c + a_j)\right)$$

$$= \sum_{i=1}^n \sum_{j=1}^n \frac{(c + a_i) w_i (c + a_j) w_j}{a_i + a_j + c}$$

$$= \sum_{i=1}^n \sum_{j=1}^n \frac{v_i v_j}{a_i + a_j + c} \qquad (\text{since } (c + a_i) w_i = v_i \text{ and } (c + a_j) w_j = v_j).$$

Hence,

$$\sum_{i=1}^n \sum_{j=1}^n \frac{a_i a_j w_i w_j}{a_i + a_j + c} \geq -c \left(\sum_{i=1}^n w_i\right)^2 \text{ holds if and only if } \sum_{i=1}^n \sum_{j=1}^n \frac{v_i v_j}{a_i + a_j + c} \geq 0. \tag{9}$$

Also, clearly,

$$v_1 = v_2 = ... = v_n = 0 \text{ holds if and only if } (c + a_1) w_1 = (c + a_2) w_2 = ... = (c + a_n) w_n = 0. \tag{10}$$

By Corollary 9, the inequality $\sum_{i=1}^n \sum_{j=1}^n \frac{v_i v_j}{a_i + a_j + c} \geq 0$ holds, with equality if and only if $v_1 = v_2 = ... = v_n = 0$. According to (9) and (10), this means that $\sum_{i=1}^n \sum_{j=1}^n \frac{a_i a_j w_i w_j}{a_i + a_j + c} \geq -c \left(\sum_{i=1}^n w_i\right)^2$, with equality if and only if $(c + a_1) w_1 = (c + a_2) w_2 = ... = (c + a_n) w_n = 0$. Thus, Corollary 10 is proven.

The problem follows from Corollary 10 (applied to $c = -1$ and $a_i = i$).