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SYSTEMS ANALYSIS

STATE EQUATIONS AND EQUIVALENT TRANSFORMATIONS FOR TIMED PETRI NETS

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INTRODUCTION

Petri nets provide a formal model of behavior of asynchronous concurrent systems. The simplicity and the high level of abstraction of the Petri net model have led to numerous applications, including design of distributed data processing systems, analysis of industrial and organizational systems, and other areas. The basic elements of the theory of Petri nets are presented in [1-4].

Problems that involve modeling system behavior over time have necessitated an extension of the original definition of a net, which does not include time. The properties of timed Petri nets have been studied in [5-8]. Applications of timed Petri nets and other extensions for the design of flexible computer-aided manufacturing systems are considered in [9]. Net models of systems and processes are constructed in [1, 2, 9] using compositions of Petri nets with external (input and output) places. Controllability of such nets has been studied in [10].

The purpose of the present study is to develop algebraic methods of equivalent transformation of timed Petri nets with external input and output places. The approach relies on the mathematical description of net behavior and transfer functions.

Different definitions of timed Petri nets are available, and we accordingly discuss the featured of the relevant class of timed nets. As in [5, 6], we consider time delays associated with transitions in the net. This technique of introducing time is useful for system modeling: transitions in the net represent actions in the original system. And places correspond to conditions (resources) required to start the actions. As in [7] we assume that the net time is discrete, and time delays are represented by positive integers. Generally, a minimum time interval (a time step) can be established as the unit of time measurement in the system being modeled.

Most studies of timed nets [5, 7, 9] assume that an active transition is blocked for the duration of its firing time. Thus, each transition is implicitly linked with some resource, of which only a single instance is a available in the net. Modeling of real system necessitates duplication of transitions, increasing the complexity of the net. Model size has been reduced in [8] by introducing nets that allow repeated starting of active transitions.

In the present article, we introduce and analyze a more general class of timed Petri nets, which allow repeated starting of active transitions not only in sequential time instants, but also simultaneously; our timed Petri nets also allow multiple firing of transmissions. A transition is thus regarded as some abstract (virtual) action. The starting of specific actions in the modeled system is determined by the available resources, as represented by the marking of input places in the corresponding transition. Drawing on the analogy with queueing systems, we say that each transition is a many-channel server with an unlimited number of channels. Place markings specify the actual restrictions on the number of channels. Elements of the theory of ensembles are used in [1] for mathematical description of multiple arcs.

Figure 1 is a diagram illustrating the capabilities of traditional timed nets (with single-channel transitions) and timed nets with multichannel transitions: the k single-channel transitions shown in Fig. 1a are simulated by the one multichannel transition of the net shown in Fig. 1b. The restriction on the number of channels is provided by the tones in place $p3$.

Behavioral types of equivalence associated with reachable marking sets and free languages have been studied for Petri nets [1, 2]. It has been shown in [1] that the corresponding general algorithmic problems are unsolvable. Net reduction methods

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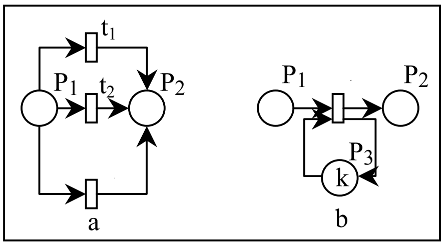


Fig. 1. Graphic representation of times Petri nets: a) net with s single-channel transition, b) a net with a multichannel transition

[3, 4] can be treated as a specific form of equivalent transformation. In this setting, equivalent nets are nets with the same set of behavioral properties (boundedness, liveness, safety).

In the present article, we introduce functional equivalence of Petri nets. A timed Petri net with external input and output places is treated as a "black box" that transforms the input stream of tokens into some output stream. This mapping of token streams is the transfer function of the net. Nets with identical transfer functions are (functionally) equivalent.

The behavior and equivalence of timed Petri nets with multichannel transitions are investigated using the algebraic approach originally proposed in [11] for nontimed nets. Contrary to [11], the state equation describing the behavior of a timed net contains additional operations, such as delay, whole division, and operations of multivalued logic. The algebra with these additional operations has specific laws, which correspond to structural transformations of nets.

Section 1 provides a formal definition of the class of timed nets. Section 2 constructs the state eguation, which constitutes a complete mathematical description of the dynamics of a timed Petri net with multichannel transitions. Sections 3 and 4 apply the state equation to find the transfer function and to investigate functional equivalence of nets with external input and output places. Section 5 proposes a method of equivalent transformations of nets based on algebraic transformations of the transfer function equations.

1. MAIN CONCEPTS AND DEFINITIONS

A timed Petri net is a bipartite directed loaded graph on which a dynamical process is defined.

Definition 1. The graph of a timed net is the 5-tuple $G = (P,T,F,W,D)$, where $P = \{p\}$ is the finite set of places, $T = \{t\}$ is a finite set of transitions. Places and transitions are the nodes of the graph. The node adjacency relation $F \subseteq (P \times T) \cup (T \times P)$ defines the arcs in the graph. The mapping $W : F \to \{1,2,...\}$ defines the multiplicity of the arcs, and the mapping $D : T \to \{1,2,...\}$ specifies the transitions firing times.

The multiplicity of the arcs $(p,t) \in F, (t,p) \in F$ and the firing times of the transitions $t \in T$ are denoted by $w_{p.t.}, w_{t.p.}, d_t$. We also use the following notation:

$^\bullet p = \{t | (t,p) \in F|\}$ is the set of input transitions of place p;

$p \bullet = \{t | (p,t) \in F|\}$ is the set of output transitions of place p;

$^\bullet t = \{p | (p,t) \in F|\}$ is the set of input places of transition t;

$t^\bullet = \{p | (t,p) \in F|\}$ is the set of output places of transition t;

Definition 2. The marking of a net is the mapping $M : P \to \{0,1,...\}$ that defines the allocation of dynamical elements (tokens) to places in the net.

The net functions in discrete time. The time is divided into steps, which are indexed by the integer variable $\tau = 0,1,2,...$. The marking of the net in step τ is the set of numbers $M(\tau) = \{\mu_p(\tau) \mid p \in P\}$. The graph G together with the initial marking $M_0 = M(0)$ constitutes a timed Petri net. In general, the initial states of the transitions also must be specified. A formal definition of a timed Petri net is therefore given in the next section, after introducing the parameters that completely describe the state of place and transitions in the current time step.

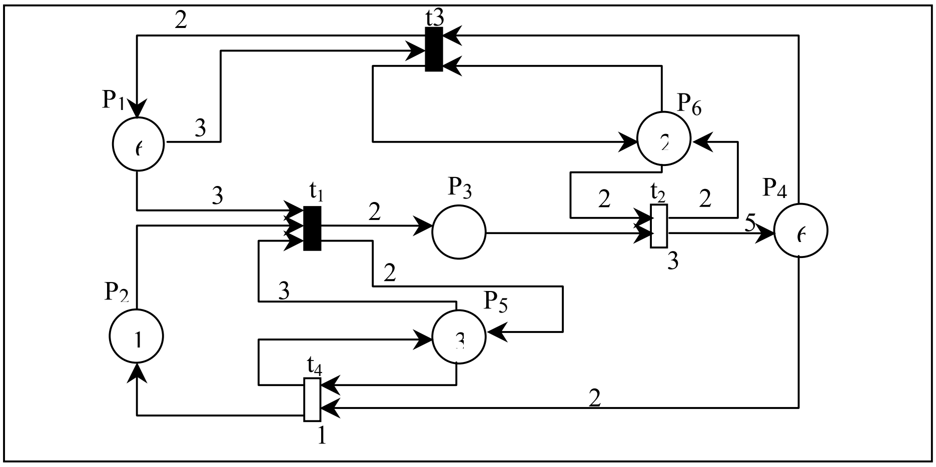


Fig. 2. Timed Petri net N_1.

The operation of a net involves changing its marking through the firing of transitions. The transition firing rules are introduced in Definition 3. Note that the marking of all places remains unchanged during each time step; the marking is modified only at the instants when time steps succeed one another. Events that occur at time change points are conventionally assigned to the next time step.

Definition 3. The transition firing rules in a timed net are formulated as follows.

1. The transition $t \in T$ is excited if the marking of all its input places $p \in {}^{\bullet}t$ is not less than the multiplicity $w_{p.t.}$ of the corresponding arcs.

2. The firing of transition $t \in T$ involves starting the transition and finishing it after d_t time steps. When transition t is started, $w_{p.t.}$ tokens are extracted from all its input places $p \in {}^{\bullet}t$. When transition t is finished, $w_{p.t.}$ markers are added to each of its output places $p \in t^{\bullet}$.

3. Transitions start and finish at time change points in the following way: first all previously started transitions for which the firing time has expired are finished; then the excitation conditions are determined and the excited transitions are started.

4. There is an *a priori* unlimited number of instances (channels) of each transition. An arbitrary ensemble of excited transitions that do not produce negative markings is started.

The simple transition firing rules require some additional clarification. The starting of an arbitrary ensemble of excited states involves not only some set of transitions, but also a certain number of instances of each excited transition. It is therefore expedient to define the excitation condition not as a binary variable 12, but as a nonnegative integer equal to the number of excited instances of the transition. In accordance with the above transition firing rules, the number of started instances (starting multiplicity) does not exceed the number of excited instances (excitation multiplicity). The operation of a net can be described by a transition-starting time chart [7].

In general a timed Petri net is a nondeterministic system. This is attributable to two factors. First, the transition firing rules do not require instantaneous starting of the excited transitions. A transition may remain in an excited state for an unlimited time, starting in an arbitrary time step as long as the excitation condition has not been removed.

Definition 4. A net in which a maximal ensemble of excited transitions are started in every time step $\tau = 0, 1, 2, \ldots$ is called synchronous.

Note that the maximal ensemble is not necessarily the greatest. Maximality is understood in the sense that starting one more excited instance of any transition produces negative markings. The existence of several maximal ensembles of starting transitions is associated with conflicts in nets. The presence of conflicts is a second reason for indeterminate behavior of timed nets.

The subclass of conflict-free timed nets is the simplest to study. However, as shown in [10], conflict-freedom is a complex behavioral property, which is difficult to define even for basic Petri nets. A sufficient condition of conflict-freedom [10] is structural conflict-freedom of the net.

Definition 5. A net is structurally conflict-free if each place has at most one output transition: $\forall_p \in P: \left|p^{\bullet}\right| \leq 1$.

A structurally conflict-free synchronous timed net is a deterministic system, because in such a net the ensemble of starting transition is always identical with the ensemble of excited transitions.

In Secs. 2 and 3 below we consider timed nets of a general form. The main results of Secs. 4 and 5 are obtained for the subclass of structurally conflict-free synchronous nets.

2. THE STATE EQUATION OF A TIMED NET

In basic Petri nets, a transition fires instantaneously. Therefore the current state of the net is uniquely determined by its marking. The state of a timed net includes the state of the places (the marking) and the state of the transitions. Most studies of timed nets abstract from transition states and consider only the marking as the most essential component of the state of the net. Thus, Chretienne [7] uses equations that link the net marking with vectors counting transition stars and finishes. The counting vectors are estimated approximately. To study the behavior of a timed net, we need variables that fully define state.

Some studies of timed nets [5,6] have noted that the current state of a transition can be described by the time remaining to its completion. However, there are no full format descriptions of net dynamics using these variables.

In the present article, we describe the current state of a transition at an arbitrary time instant τ by the past history of its starts in the time interval $[\tau - d_t + 1, \tau]$. Since time is represented by integers, we can consider a start history consisting of d_t values for each transition.

Definition 6. The state of a timed net in time step τ is the pair $S(\tau) = (S^P, S^T(\tau))$ where $S^P(\tau)$ is the state of the places and $S^T(\tau)$ is the state of the transitions. Here

$$S^P(\tau) = M(\tau) = \{\mu_p(\tau) \mid p \in P\},$$

$$S^T(\tau) = U(\tau) = \{u_t(\tau - \Theta) \mid t \in T), \Theta = \overline{0, d_t - 1}\}, \qquad (1)$$

where $u_t(v)$ is the number of instances (channels) of transition t started in time step v. The transition start history $U(\tau)$ can be stored in an $n \times k$ matrix, where $|T|$, $k = \max(d_t)$. For each specific transition t, d_t matrix elements are used.

Definition 7. The timed Petri net N is the pair $N = (G, S_0)$, where G is the net graph and $S_0 = S(0)$ is the initial state of the net.

In the previous section we considered only the initial marking M_0 and assumed that all transitions were initially passive, i.e., $U_0 = 0$. In general, we have to consider nonzero values of U_0. By definition (1), the initial transition state U_0, and also the state $U(\tau)$ up to time $\tau = k$ are assigned negative time values in the transition start history. Therefore, on an abstract level, we can consider transitions started "before the initial time instant." This ensures a more regular description of the net, without clashing with natural view of process dynamics, because the choice of zero time instant is often arbitrary.

To construct the state equation, we introduce a number of auxiliary variables: $v_t(\tau)$ is the number of instances of transition τ excited in step τ; $\mu_p(\tau)$ is the intermediate marking of place τ at the instant when time step $\tau - 1$ changes to τ (this marking is obtained when the previously started transitions finish).

Let us illustrate these concepts by a specific example. Figure 2 is timed Petri net N_1 with 6 places and 4 transitions. The firing times are shown for each transition. The multiplicity is given for each arc. The unlabeled arcs have multiplicity 1. The numbers inside places define the initial marking of the net. Transitions t_1 and t_3 are active in the initial state: $u(t_1, 0) = 1$, $u(t_3 - 1) = 3$, $u(t_3, 0) = 3$. They are therefore painted solid black.

Figure 3 is the time chart showing the operation of the net N_1. Note that the process shown in Fig.3 is one of many possible processes that satisfy the transition firing rules (Definition 3). For each transition, the vertical axis gives the maximum number of channels that can be simultaneously active on the time interval shown. The tables below the chart the dynamics of changes in μ_p', v_f, u_r, μ_p over time. The process is cyclic, because $S(10) = S(0)$. The net behavior is asynchronous. Already in time step $\tau = 1$ the starting transition ensemble $(1, 1, 0, 1)$ is not maximal. The maximal ensembles in this time step are $(1,1,3,1)$, $(1,0,5,0)$, and $(0,1,2,2)$, the first two of which are also the greatest. The starting transition ensemble is maximal only in steps $\tau = 3$ and $\tau = 13$.

In some cases, it is expedient to consider the action of outside forces on the net in the form of a stream of tokens arriving from the outside in some places in the net. We denote by $\alpha_p{}^{\tau}$ the tokens that reach place from the outside in time step $\tau - 1$ and take part in the excitation of transitions in time step τ.

Assertion 1. The dynamics of the timed Petri net N in accordance with the transition firing rules of Definition 3 is completely described by the system of equations and inequalities (2):

$$\begin{cases} \mu'(\tau) = \mu_p(\tau-1) + \sum_{t \in {}^{\bullet}p} w_{t,p} \cdot u_t(\tau - d_t) + \alpha_p^{\tau}, & (2.1) \\ \mu_p(\tau) = \mu_p - \sum_{t \in p^{\bullet}} w_{t,p} \cdot u_t(\tau), & (2.2) \\ \mu_p(\tau) \geq 0, 0 \leq u_t(\tau) \leq \upsilon_t(\tau), & (2.3) \\ \upsilon_t(\tau) = \mathop{\&}_{q \in {}^{\bullet}t} \mu_q(\tau) / w_{q,t}, p \in P, t \in T, & (2.4) \\ S(0) = S_0, \tau = 1,2,..., & (2.4) \end{cases}$$

Here the conjunction $(\&)$ is interpreted as in multivalued logic [13]: $x \,\&\, y = \min(x, y)$. Division is whole division (the whole part of the quotient). Following [12], we call system (2) the state equation of a timed net.

To prove the assertion, we will show that system (2) corresponds to the transition firig rules of Definition 3. Indeed, $v_t(\tau)$ in Eq. (2.4) is nonzero only if the marking of all input places of the transition is not less than the multiplicity of the corresponding arcs: $\mu_q{}'/w_{q,t} \geq 1 \leftrightarrow \mu_q{}'(\tau) \geq w_{q,t'}$, This corresponds to rule 1. Equations (2.1) and (2.2) describe transition starts and finishes by rule 2. The use intermediate markings $\mu_q{}'(\tau)$ in excitation conditions (2.4) enables us to determine the action at the time change point in accordance with rule 3. The system of inequalities (2.3) ensures that rule 4 is satisfied.

The initial definition of net dynamics does suggest what transitions are started from the ensemble of excited transitions. This corresponds to the existence of different solutions of the inequalities (2) and is determined by the specific formulation of the problem for the system being modeled. For synchronous nets the inequalities should be solved so that for every $u_t(\tau) < v_t(\tau)$ the solution $u_t(\tau)+1$ produces negative markings.

The case when the set of input places ${}^{\bullet}t$ for some transition $t \in T$ is empty requires special discussion. For such a transition, the value of the conjunction in Eq. (2.4) is the symbol ω, which corresponds to an unbounded excitation multiplicity. In inequalities (2.3) we take an arbitrary finite value for $u_t(\tau)$. For synchronous nets we ignore transitions with an empty set of input places (sources), because maximality of the ensemble of starting transitions requires starting an unbounded number of transition instances.

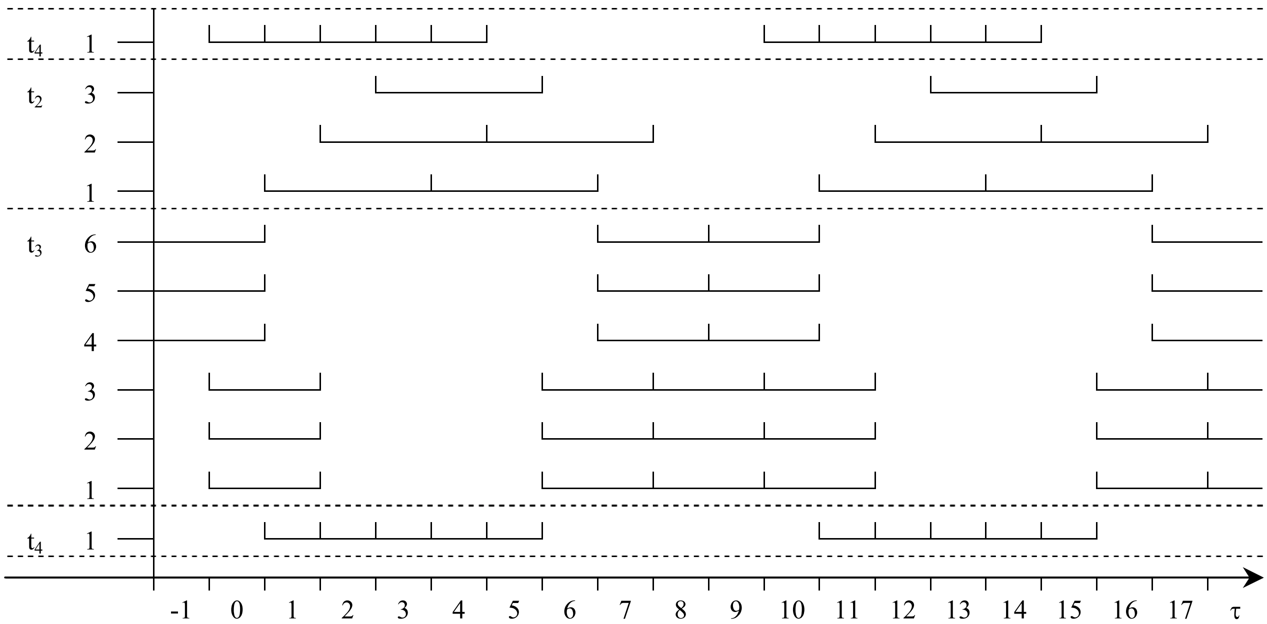
For structurally conflict-free synchronous nets system (2) is essentially simplified. Since $u_t(\tau) = v_t(\tau)$, the inequalities drop out from the system, and the behavior of the net is completely described by the system of recurrence equations.3

3. FUNCTIONAL EQUIVALENCE OF NETS

The rest of the article focuses on timed Petri with external input and output places. Nets with external input and output places have been used in [1, 9] to model complex multicomponent systems. A composition of such nets is constructed by superimposing their external places. The controllability problem for such nets, i.e., the problem of taking the net to a specified final state, is solved in [10]. In the article we investigate the functional dependence of the stream of tokens delivered to output places on the stream of tokens reaching the input places. The net is treated as a dynamical system that transforms the input stream of tokens info some output stream.

Definition 8. A net with input and output places is the triple $Z = (N, X, Y)$, where N is a timed Petri net, $X \subseteq P$ is the set of input places, and $Y \subseteq P$ is the set of output places, $X \cap Y = \varnothing$. Input places have no entering arcs and output places have no leaving arcs: $\forall_x \in X: {}^{\bullet}x = \varnothing, \forall_y \in Y : y^{\bullet} = \varnothing$. The places in the set $R = P/(X \cup Y)$ are called internal places.

We assume that outside actions are applied only to the input places of the net: $\forall_p \notin X, \forall_\tau > 0 : \alpha_p^\tau = 0$.



		-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	τ
μ'	p_1			12	15	12	9	6	6	3	6	9	12	12	15	12	9	6	6	3	6
	p_2			1	1	1	1	1	2	2	2	2	2	1	1	1	1	1	2	2	2
	p_3			1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0
	p_4			6	4	2	5	8	11	13	15	12	9	6	4	2	5	8	11	13	15
	p_5			5	5	5	5	5	6	6	6	6	6	5	5	5	5	5	6	6	6
	p_6			5	6	4	4	4	4	3	5	5	5	5	6	4	4	4	4	3	5
v	t_1			1	1	1	1	1	2	1	2	2	2	1	1	1	1	1	2	1	2
	t_2			1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0
	t_3			5	4	2	4	4	4	3	5	5	5	5	4	2	4	4	4	3	5
	t_4			2	2	1	2	2	3	3	3	3	3	2	2	1	2	2	3	3	3
u	t_1	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0
	t_2	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0
	t_3	3	3	0	0	0	0	0	3	3	3	3	3	0	0	0	0	0	3	3	3
	t_4	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0
μ	p_1		6	9	12	9	6	6	3	0	3	6	6	9	12	9	6	6	3	0	3
	p_2		1	0	0	0	0	1	2	2	2	2	1	0	0	0	0	1	2	2	2
	p_3		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	p_4		6	4	2	2	3	6	8	10	12	9	6	4	2	0	3	6	8	10	12
	p_5		3	0	0	0	0	3	6	6	6	6	3	0	0	0	0	3	6	6	6
	p_6		2	3	4	4	2	2	1	0	2	2	2	3	4	2	2	2	1	0	2

Fig. 3. Time chart of the dynamics of the net N_1.

Definition 9. An input sequence α of the net Z is a set of nonnegative integers $\alpha = \{\alpha_x^{\tau} \mid x \in X, \tau = 1,2...\}$.An output sequence ϕ of the net Z is a sequence of tokens $\{\Phi_x^{\tau} \mid y \in Y, \tau = 1,2...\}$, that are delivered to its output places as a result of the operation of the net N.

Note that the output sequence ϕ is determined by the input sequence α and the state equation (2) of the net. However, because of nondeterministic operation of the net, a fixed input sequence α corresponds to a whole set $\Phi = \{\phi\}$ of output sequences.

Definition 10. The mapping f_z of the set of input sequences $\{\alpha\}$ of the net z on the set of its output sequences $\{\Phi\}$ is called the transfer function of the net, $\Phi = f_z(\alpha)$.

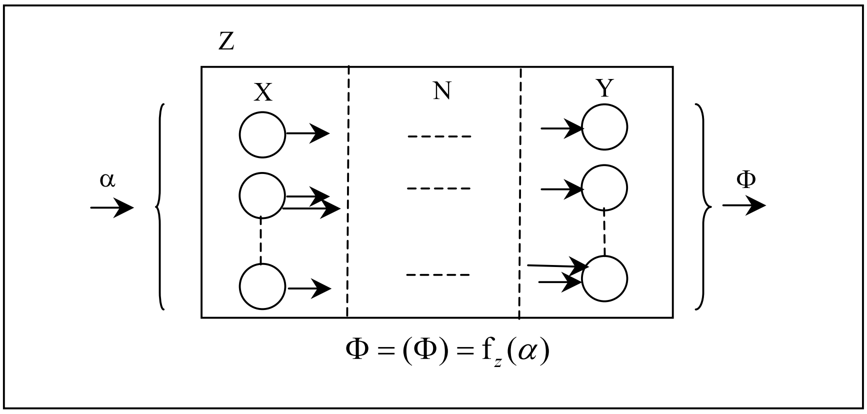


Fig. 4. Net with external input and output places

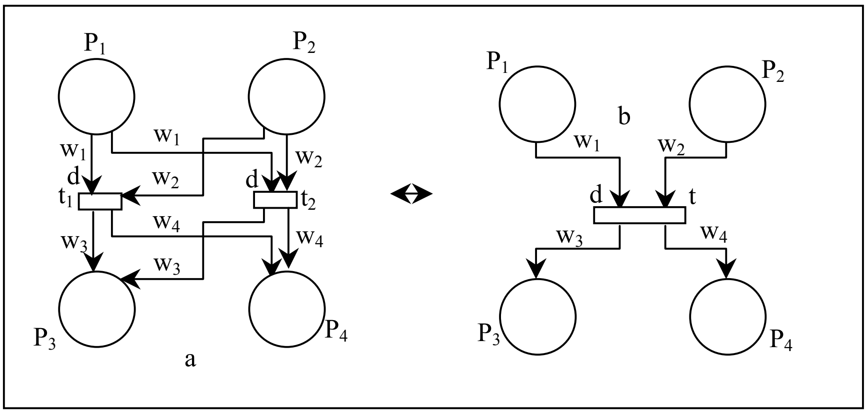


Fig. 5. Example of functionally equivalent nets.

In Fig. 4 the net z with input and output places is schematically shown as a "black box" that transforms the input sequence of tokens α into a set of admissible output sequences Φ. For interaction with the outside world, only the subsets X and Y of net places are used. For an outside observer, the behavior of the net is completely characterized by its transfer function f_z.

Transfer functions are compared only for comparable nets, i.e., nets for which the sets of input places and the sets of output places respectively are of equal cardinality. We also assume that a one-to-one correspondence is defined between the external places of the nets, e.g., by some indexing.

Definition 11. The nets z and z' are called functionally equivalent (notation, $z \equiv z'$) if the sets of their output sequences are identical for any input sequence $\alpha : f_z(\alpha) = f_{z'}(\alpha)$.

This is the strongest definition of functional equivalence. Weaker types of functional equivalence in relation to a particular type of input sequences and also in relation to specific time instants fall outside the scope of our article.

The proposed strong definition of functional equivalence enables us to replace an arbitrary net z with an equivalent net z' in any context of outside actions. Moreover, decomposing some timed net N into subnets with input and output places and replacing some subnets with their equivalents, we obtain a transformation of the original net that completely preserves the characteristics of the untouched elements. General issues of decomposition of Petri nets into subnets with input and output places are not considered in this article.

The transfer function f_z of the net z is implicitly specified by its state equation (2). The marking of output places can be ignored, because it does not participate in the excitation of transitions. The corresponding equations should be replaced with relationships that define the output sequence of the net:

$$\Phi_y^\tau = \sum_{t \in {}^\bullet Y} w_{t,y} \cdot u_t(\tau - d_t), y \in Y .$$

$$\begin{cases} \mu_1(\tau) = \mu_1(\tau-1) + \alpha_1^\tau, \\ \mu_1(\tau) = \mu_1(\tau) - w_1 \cdot u_1(\tau) - w_1 \cdot u_2(\tau), \mu_1(\tau) \ge 0, \\ \mu_2(\tau) = \mu_2(\tau-1) + \alpha_2^\tau, \\ \mu_2(\tau) = \mu_2(\tau) - w_2 \cdot u_1(\tau) - w_2 \cdot u_2(\tau), \mu_2(\tau) \ge 0, \quad (3) \\ \phi_3^\tau = w_3 \cdot u_1(\tau - d) + w_3 \cdot u_2(\tau - d), \\ \phi_4^\tau = w_4 \cdot u_1(\tau - d) + w_4 \cdot u_2(\tau - d), \\ \upsilon_1(\tau) = \upsilon_2(\tau) = (\mu_1(\tau)/w_1) \,\&\, (\mu_2(\tau)/w_2), \\ 0 \le u_1(\tau) \le \upsilon_1(\tau), 0 \le u_2(\tau) \le \upsilon_2(\tau) \end{cases}$$

Equivalent transformations of the state equation correspond to structural transformations of the net preserving its transfer function. This approach can be applied to generate a whole set of transformations of simple nets. The purpose of the transformations is to reduce the net size.

For instance, for the net shown in Fig. 5a, $X = \{p_1, p_2\}, Y = \{p_3, p_4\}$. The transfer function is described by the following system of equations: Denote $u(\tau) = u_1(\tau) + u_2(\tau)$ and $v(\tau) = v_1(\tau) + v_2(\tau)$. A necessary and sufficient condition of nonnegativity of $\mu_1(\tau)$ and $\mu_1(\tau)$ is the inequality $0 \le u(\tau) \le v(\tau)$. System (3) thus can be replaced with the equivalent system (4):

$$\begin{cases} \mu_1'(\tau) = \mu_1(\tau-1) + \alpha_1^\tau, \\ \mu_1(\tau) = \mu_1(\tau) - w_1 \cdot u(\tau), \mu_1(\tau) \ge 0, \\ \mu_2'(\tau) = \mu_2(\tau-1) + \alpha_2^\tau, \\ \mu_2(\tau) = \mu_2(\tau) - w_2 \cdot u(\tau), \mu_2(\tau) \ge 0, \quad (4) \\ \phi_3^\tau = w_3 \cdot u(\tau - d), \\ \phi_4^\tau = w_4 \cdot u(\tau - d), \\ \upsilon(\tau) = (\mu_1'(\tau)/w_1) \,\&\, (\mu_2'(\tau)/w_2), \\ 0 \le u(\tau) \le \upsilon(\tau) \end{cases}$$

System (4) corresponds to the net shown in Fig. 5b. The nets in Figs. 5A and 5b are thus functionally equivalent. The transformation has removed four arcs and one transition.

For general timed nets, the application of this approach involves considerable difficulties, because the state equation describes the transfer function of the net in an implicit form, and in addition to the variables α and ϕ characterizing the functional dependence it also contains a number of auxiliary variables. The main results associated with equivalent transformations will be obtained for the subclass of structurally conflict – free synchronous nets.

4. TRANSFER FUNCTION OF A STRUCTURALLY CONFLIST – FREE NET

In what follows we focus on synchronous structurally conflict-free timed Petri nets with input and output places. The behavior of such nets is deterministic, and a fixed input sequence α produces a unique output sequence $\phi = f_{z'}(\alpha)$.

Define $\beta_p(\tau)$ or $(\beta_p{}^{\tau})$ as the total number of tokens arriving in place p up to time τ. This variable makes it possible to express the transfer function of the net (Theorem 1 below) from the state equation (2) by eliminating the variables μ, μ', u that characterize the state of the internal net elements. By definition, the sequence $\beta_p{}^{\tau}, \tau = 1,2,...$, is nondecreasing for $\forall p \in P$. In accordance with the state equation (2), the values of $\beta_p{}^{\tau}$ for input, internal, and output places are defined by relationships (5)-(7) respectively:

$$\beta_x^{\tau} = \sum_{\theta=1}^{\tau} \alpha_x^{\theta}, x \in X. \tag{5}$$

$$\beta_r^{\tau} = \sum_{\theta=1}^{\tau} \sum_{t \in {}^*r} w_{t,r}\ u_t(\theta - d_t), r \in R, \tag{6}$$

$$\beta_y^{\tau} = \sum_{\theta=1}^{\tau} \Phi_y^{\theta} = \sum_{\theta=1}^{\tau} \sum_{t \in {}^*y} w_{t,y}\ u_t(\theta - d_t), y \in Y. \tag{7}$$

We introduce a notation for the sequences $\beta_p{}^{\tau}$ for input and output places:

$$\beta^X = \{\beta_x^{\tau} \mid x \in X, \tau = 1,2,...)\}, \beta^Y = \{\beta_y^{\tau} \mid y \in Y, \tau = 1,2,...)\}. \tag{8}$$

Note that β^X and β^Y are sequences of partial sums of the sequences α and ϕ, respectively. Therefore, the transfer function of the net z may be regarded as a mapping of the sequences β^X into the sequences $\beta^Y : \beta^Y = g_Z(\beta^X)$. Theorem 1 defines the system of equations specifying the transfer function of a net relative to partial sums of input and output sequences. Assertion 2 establishes that the use of the transfer function g_Z is invariant to the initial definition of functional equivalence.

Assertion 2. $Z \equiv Z'$ if and only if for any nondecreasing sequence β^X we have $g_Z(\beta^X) = g_{Z'}(\beta^X)$.

The assertion is proved using the one-to-one correspondence between the sequences α and β^X and between the sequences ϕ and β^Y. The direct mapping is defined by relationships (5) and (7); the inverse mapping is defined by

$$\alpha_x^1 = \beta_x^1; \alpha_x^{\tau} = \beta_x^{\tau} - \beta_x^{\tau-1}, x \in X, \tau = 2,3,...$$

$$\Phi_y^1 = \beta_y^1; \Phi_y^{\tau} = \beta_y^{\tau} - \beta_y^{\tau-1}, y \in Y, \tau = 2,3,...$$

THEOREM 1. The transfer function of the structurally conflict-free synchronous timed Petri net Z is described by the system of equations

$$\beta_p(\tau) = \mu_p^0 + \sum_{t \in \bullet p} w_{t,p} \cdot \underset{q \in \bullet t}{\&} (\beta_q(\tau - d_t) / w_{q,t}) + \delta_p(\tau), p \in R \cup Y, \tag{9}$$

where $\delta_p(\tau)$ is equal to the total inflow of tokens reaching place p by time τ from transitions active in the initial state. Note that $\delta_p(\tau) = const$ for $\tau > k$. If the transitions are passive in the initial state, then $\delta_p(\tau) = 0$; otherwise

$$\delta_p(\tau) = \sum_{t \in {}^\bullet p} w_{t,p} \cdot \sum_{\theta = -d_t + 1}^{\min(\tau - d_t, 0)} u_t^{\theta} \tag{10}$$

Proof. Construct the state equation of the net Z ;

$$\begin{cases} \mu'_x(\tau) = \mu_x(\tau-1) + \alpha_x^\tau, x \in X \\ \mu'_p(\tau) = \mu_p(\tau-1) + \sum_{t \in {}^\bullet p} w_{t,p} \cdot u_t(\tau - d_t), p \in R \cup Y \\ u_t(\tau) = \underset{q \in {}^\bullet t}{\&} (\mu'_q(\tau) / w_{q,t}), t \in T \\ \mu_p(\tau) = \mu'_p(\tau) - w_{p,g} \cdot u_g(\tau), p \in X \cup R, p^\bullet = \{в\} \\ \mu_y(\tau) = \mu'_y(\tau), y \in Y. \end{cases} \quad (11)$$

Express from (11) the value of $\beta_p{}^\tau$ for internal and output places using (6), (7):

$$\beta_p(\tau) = \mu_p^0 + \sum_{\theta=1}^{\tau} \sum_{t \in {}^\bullet p} w_{t,p} \cdot u_t(\theta - d_t), p \in R \cup Y. \quad (12)$$

Since $w_{t,p}$ is independent of τ, we change the order of the summation indices in (12) and then separate the time interval $[1, d_t]$ where all the initially active instances of the transition t finish:

$$\beta_p(\tau) = \mu_p^0 + \sum_{\theta=1} w_{t,p} \sum_{t \in {}^\bullet p}^{d_t} u_t(\theta - d_t) + \sum_{t \in {}^\bullet p} w_{t,p} \cdot \sum_{\theta = d_t + 1} u_t(\theta - d_t) =$$

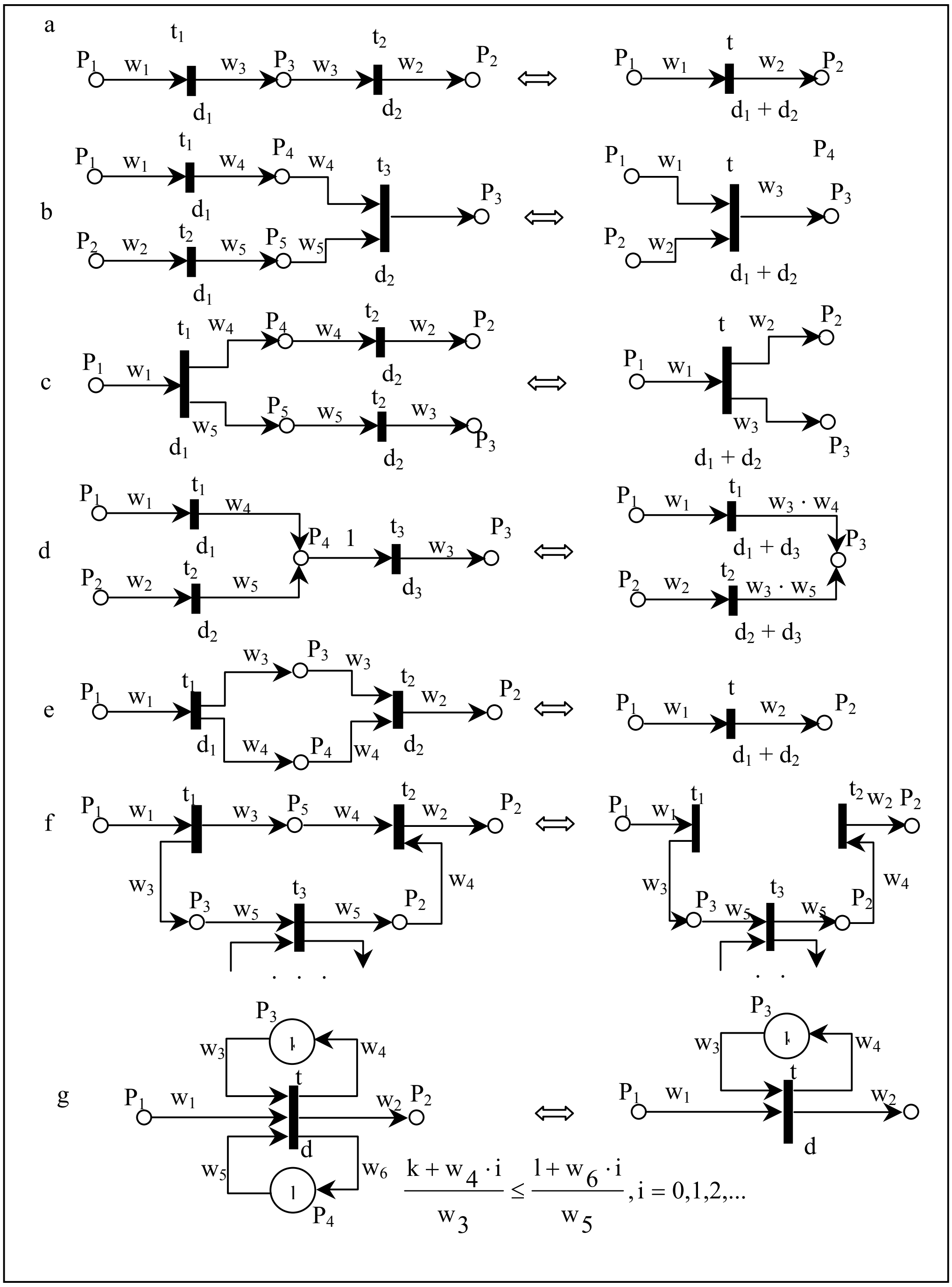


Fig. 6. Equivalent transformations of nets.

$$\mu_p^0+\sum_{t\in^\bullet p} w_{t,p}\cdot\sum_{\theta=-d_t+1}^{\min(0,\tau-d_t)} u_t+\sum_{t\in^\bullet p} w_{t,p}\cdot\sum_{\theta=d_t+1}^{\tau} u_t(\theta-d_t)=\mu_p^0+\sum_{t\in^\bullet p} w_{t,p}\cdot\sum_{\theta=d_t+1}^{\tau} u_t(\theta-d_t)+\delta_p(\tau). \qquad (13)$$

To prove the theorem, we have to express the sum of $u_t(\theta-d_t)$ over the index θ in (13) in terms of the values $\beta_p(\tau)$ of the input places of transition t. The value of this sum in time steps $\tau<d_t+1\ \tau$ is taken equal to zero.

From system (11) we write the equation of the marking of an arbitrary input or internal place q and substitute the corresponding $\mu_q{}'(\tau)$ in this equation:

$$\mu_q(\tau)=\mu_q(\tau-1)+\sum_{t\in^\bullet p} w_{t,p}\cdot u_t(\tau-d_t)-w_{q,g}\cdot u_g(\tau), p\in X\cup R. \qquad (14)$$

Successively substituting in (14) the markings from the preceding time instants down to zero, we obtain $\mu_q(\tau)=\beta_q(\tau)-w_{q,g}\cdot\sum_{\theta=1}^{\tau} u_g(\theta).$ Now using the relationship $\mu_q^{'}(\tau)=\mu_q(\tau)-w_{q,g}\cdot u_g(\tau),$ we obtain

$$\mu_q^{'}(\tau)=\beta_q(\tau)-w_{q,g}\cdot\sum_{\theta=1}^{\tau} u_g(\theta). \qquad (15)$$

From (15) we find μ' in time step $\tau-d_g$:

$$\mu_q^{'}(\tau-d_g)=\beta_q(\tau-d_g)-w_{q,g}\cdot\sum_{\theta=d_{t+}1}^{\tau-1} u_g(\theta-d_g). \qquad (16)$$

We use the value of $\mu'(\tau-d_g)$ from (16) to find $u_g(\tau-d_g)$ in accordance with (11):

$$u_g(\tau-d_g)=\underset{q\in^\bullet в}{\&}(\mu_q^{'}(\tau-d_g)/w_{q,g})=\underset{q\in^\bullet в}{\&}([\beta_q(\tau-d_g)-w_{q,g}\cdot\sum_{\theta=d_g+1}^{\tau-1} u_g(\theta-d_g)]/w_{q,g})=$$

$$\underset{q\in^\bullet в}{\&}(\beta_q(\tau-d_g)/w_{q,g}-\sum_{\theta=d_g+1}^{\tau-1} u_g(\theta-d_g). \qquad (17)$$

Moving the sum to the left-hand side, we obtain

$$\sum_{\theta=d_g+1}^{\tau-1} u_g(\theta-d_g)=\underset{q\in^\bullet в}{\&}(\beta_q(\tau-d_g)/w_{q,g}. \qquad (18)$$

Substituting (18) in Eq. (13), we obtain Eq. (9). Q.E.D.

We thus have a system of equations that describe the transfer function of a structurally conflict-free net. For acyclic nets, the transfer function can be explicitly expressed in terms of the sequence β^X of its input places.

5. EQUIVALENT TRANSFORMATIONS OF STRUCTURALLY CONFLICT-FREE NETS

In this section we present an algebraic approach to equivalent transformations of timed nets.

As in [14], we introduce an algebraic notation for the time delay operation:

$$\begin{cases} u(\tau)\triangleright d=0, \tau<d \\ u(\tau-d), \tau\geq d. \end{cases}$$

Then system (9) is representable for an arbitrary time stem in the form (19) (we omit the time step index):

$$\beta_p = \mu_p^0 + \sum_{t\in^\bullet p} w_{t,p} \cdot \underset{q\in^\bullet t}{\&}((\beta_q \rhd d_t)/w_{q,t}), p \in R \cup Y. \qquad (19)$$

System (19) includes the variables $\beta_p \in B$ where B is the set of nondecreasing nonnegative integer – valued functions of time, and also the positive integer constants $w_{t,p}, w_{q,t}, d_t$. The variables are connected by arithmetic (+, · , /), logical $(\&)$, and temporal $\{\rhd\}$ operations. Let us define the basic laws of the algebra $A=(B,\Omega)$, with base B and the signature $\Omega = \{\cdot,\&,/,>,+\}$. The operations $\{+,\cdot,\&\}$ are associative and commutative. The multiplications operation is distributive by addition. Recall that conjunction is interpreted as the operation of taking the minimum, and division is whole division, so that in general $a/c+b/c \neq (a=b)/c$. Let $\beta_i \in B$ and $w_i, d_i = const$. Let us state the specific laws of the algebra A obtained through compositions of arithmetic, logical, and temporal operations:

$$\begin{array}{ll}
1.(\beta \rhd d_t) \rhd d_2 = \beta \rhd (d_1+d_2), & 7.\beta_1 + (\beta_2 \,\&\, \beta_3) = (\beta_1+\beta_2)\,\&\,(\beta_1+\beta_3),\\
2.(\beta_1+\beta_2) \rhd d = (\beta_1 \rhd d) + (\beta_2 \rhd d), & 8.\beta_1 \cdot (\beta_2 \,\&\, \beta_3) = (\beta_1 \cdot \beta_2)\,\&\,(\beta_1 \cdot \beta_3),\\
3.(\beta_1 \cdot \beta_2) \rhd d = (\beta_1 \rhd d)\cdot(\beta_2 \rhd d), & 9.(\beta_1 \,\&\, \beta_2)/w = (\beta_1/w)\,\&\,(\beta_2/w),\\
4.(\beta_1 \,\&\, \beta_2) \rhd d = (\beta_1 \rhd d)\,\&\,(\beta_2 \rhd d), & 10.(\beta_1/w)\,\&\,\beta_2 = (\beta_1 \,\&\, w\cdot\beta_2)/w,\\
5.\, w\cdot(\beta \rhd d) = (w\cdot\beta) \rhd d, & 11.(\beta_1/w)+\beta_2 = (\beta_1 + w\cdot\beta_2)/w,\\
6.\,(\beta \rhd d)/w = (\beta/w) \rhd d, & 12.(w\cdot\beta)/w = \beta,\\
\multicolumn{2}{c}{13.\beta/(w_1+w_2) = (\beta/w_1)/w_2.}
\end{array} \qquad (20)$$

To prove the laws (20), it suffices to apply the definitions o the operations from the set Ω and to consider separately time intervals for the delay operations [14], as well as various relationships of variables valued for the conjunction operation [13].

Assume that the set of formulas of algebra A defines the transfer function of the net Z. Then equivalent transformations of formulas by the laws (20) correspond to structural transformations of nets preserving their transfer function.

Application of this approach to the nets shown in the left hand side of Fig. (a..g) has produced smaller equivalent nets that are shown in the right-hand side of the same figure. Let us consider in more detail some transformations used in Fig.6.

Transformation a

The initial net contains three places and two transitions. Represent the transfer function of the net in the form (19):

$$\beta_3 = w_3 \cdot ((\beta_1 \rhd d_1)/w_1), \;\; \beta_2 = w_2 \cdot ((\beta_3 \rhd d_2)/w_3).$$

Substitute the first equation in the second equation and successively apply the laws (20) with numbers 5, 12, 6, 1:

$$\beta_2 = w_2 \cdot \frac{(w_3((\beta_1 \rhd d_1)/w_1) \rhd d_2}{w_3} \overset{(5)}{=} w_2 \cdot \frac{(w_3(((\beta_1 \rhd d_1)/w_1) \rhd d_2)}{w_3} \overset{(12)}{=}$$
$$= w_2 \cdot (((\beta_1 \rhd d_1)/w_1) \rhd d_2) \overset{(6)}{=} w_2 \cdot (((\beta_1 \rhd d_1) \rhd d_2)/w_1) \overset{(11)}{=} w_2 \cdot ((\beta_1 \rhd (d_1+d_2))/w_1).$$

The resulting expression has the form (19). The corresponding net contains two places and one transition.

Transformation d

$$\beta_3 = w_3(\beta_4 \rhd d_3), \beta_4) = w_4((\beta_1 \rhd d_1)/w_1) + w_5((\beta_2 \rhd d_2)/w_2).$$
$$\beta_3 = w_3([w_4((\beta_1 \rhd d_1)/w_1) + w_5((\beta_2 \rhd d_2)/w_2)] \rhd d_3) =$$
$$= w_3 w_4((\beta_1 \rhd (d_1+d_3))/w_1) + w_3 w_5((\beta_2 \rhd (d_2+d_3))/w_2).$$

Transformation g

$$\beta_2 = w_2 \cdot u, \beta_3 = k + w_4 \cdot u, \beta_4 = 1 + w_6 \cdot u,$$

$$u = \frac{b_1 \rhd d}{w_1} \& \frac{b_3 \rhd d}{w_3} \& \frac{b_4 \rhd d}{w_5}.$$

For $\frac{k + w_4 \cdot i}{w_3} < \frac{1 + w_6 \cdot i}{w_5}, \forall i > 0$, we have $\frac{b_3}{w_3} < \frac{b_4}{w_5}$, then $u = \frac{b_1 \rhd d}{w_1} \& \frac{b_3 \rhd d}{w_3}$.

Thus, the general algorithm for equivalent transformations of nets involves successive execution of three main steps.

Step 1. Construct a system of algebraic equations of the form (19) defining the transfer function of the net.

Step 2. Apply laws (20) of the algebra A to perform equivalent transformations of the system of equations.

Step 3. Reduce the resulting system to the form (19) and construct the corresponding net.

Since all the operations preserve the transfer function, the initial and the final nets are functionally equivalent. A different approach to equivalent transformations is also possible: by direct application of ready-made graphical templates corresponding**,** for instance, to the equivalent nets shown in Fig. 6.

The proposed approach can be applied to synchronous timed Petri nets of a general form by identifying and subsequently transforming structurally conflict-free subnets.

CONCLUSION

We have introduced the class of timed Petri nets with multichannel transitions that generalize the standard definitions of timed Petri nets. We have derived a full mathematical description of the process of net operation, i.e ., the state equation, which provides the basis for formal methods of the net analysis.

The concepts of transfer function and functional equivalence have been introduced for nets with input and output places. The transfer function is implicitly defined by the state equation, and equivalent transformations of the state equation correspond to structural transformations of the net that preserve its transfer function.

For the subclass of structurally conflict-free synchronous timed nets we have derived the system of equations that define the transfer function in explicit form. An equivalent transformation method has been proposed, based on the laws of a special algebra with arithmetic, logical, and temporal operations.

The results obtained in this article are applicable for formal analysis, synthesis, and minimization of net models of systems and processes.

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