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Some considerations on the Bouchaud–Cates–Ravi–Edwards model for granular flow

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Abstract

We discuss some features of the BCRE model. Under some conditions, we show that it can be understood as a mapping from a two-dimensional to a one-dimensional problem. We then propose some modifications that (a) guarantee mass conservation (which is not assured in its original form) and (b) correct undesired features that appear when there are irregularities in the surface of the static phase. We also show that a similar model can be deduced both from the principle of mass conservation (first equation) and from a simple thermodynamic argument (from which the exchange equation can be obtained). Finally, we solve the model numerically, using different velocity profiles and studying the effects of different parameters. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Since the end of 1980s, when Bak et al. used a sandpile as a paradigm for self-organized criticality [1], there has been a revival in the interest in granular materials. This topic, however, is not recent. The first studies with granular materials date back to 1773, when Coulomb first observed that this kind of matter could stand in equilibrium in piles at certain specific angles. Faraday discovered the convective instability in

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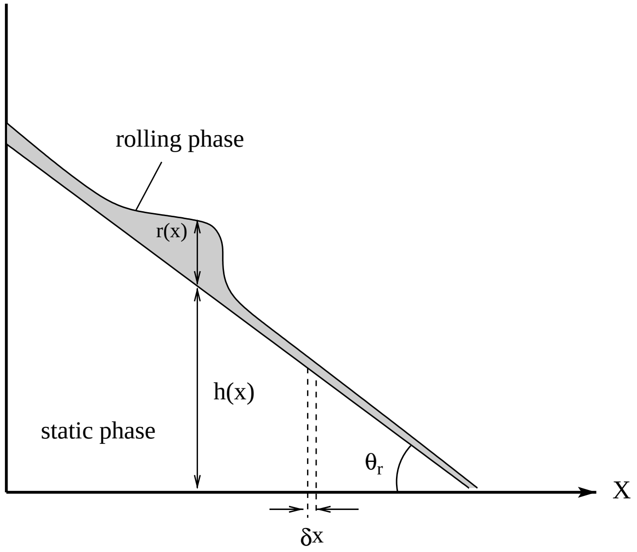

Fig. 1. In the BCRE model, a two-dimensional pile with flowing grains is divided into two “phases” (static and rolling phases). They are described by the variables h and $R = \rho r$, where h and r are the heights of the static and rolling phases, respectively.

vibrating grains and Reynolds introduced the notion of dilatancy, just to cite some examples of well-known scientists who became interested in the problem.

The study of granular materials is important since their applications in industry is wide enough to cover areas as distinct as civil construction and food transportation. The study of flows of grains can uncover the behavior of dunes, sand-storms and avalanches. But those are just a few examples. Granular materials are present everywhere in nature, and in several branches of industry. Chemical industries, pharmaceutics, mining, geology are just some additional examples of other areas where the study of grains can play an important role.

In 1994, a paper by Bouchaud et al. [2] presented a relevant model that came to be known as the BCRE model. This model was successful in describing the qualitative behavior of flowing grains, with the additional advantage of being very simple. It assumes that a two-dimensional sandpile (Fig. 1), with rolling grains on its surface, can be divided into two “phases” (a static phase h and a rolling phase R), and proposes two coupled partial differential equations to model their behavior:

$$\frac{\partial R(x,t)}{\partial t} = -\frac{\partial}{\partial x}[vR(x,t)] + \frac{\partial}{\partial x}\left[D\frac{\partial R(x,t)}{\partial x}\right] + \Gamma(R,\tilde{h}) \,, \tag{1}$$

$$\Gamma(R,\tilde{h}) = -R(x,t)\left[\gamma\frac{\partial \tilde{h}(x,t)}{\partial x} + \kappa\frac{\partial^2 \tilde{h}(x,t)}{\partial x^2}\right] = -\frac{\partial \tilde{h}(x,t)}{\partial t} \,. \tag{2}$$

The variables h and R are related to the height of the static and rolling phases, respectively, and $\tilde{h} = h + x \tan \theta_r$, where θ_r is the angle of repose. Parameters γ and κ are positive, v is the velocity profile of the rolling phase, and D is related to diffusion.

The first equation defines how the profile of the rolling phase evolves in time, and the second equation determines the profile of the static phase by setting the form of the exchange between rolling and static grains, depending on the local slope of the pile. This phenomenological model seems suitable to describe some of the properties observed in the flow of real grains. It has given important clues to how we could describe some interesting phenomena occurring in granular flow. A variety of papers used these equations to model the behavior of avalanches, stratification and flows in general [3–7].

The original model, however, is very simplified, has some problems of consistency and, as stated before, lacks a derivation from first principles or from a microscopic point of view. In this paper we address some of these points. In Section 2, we show that an expression equivalent to Eq. (1) can be obtained from the principle of mass conservation, under the assumption that the densities of the static and rolling phases are constant along the vertical direction. In Section 3, we discuss some limitations of Eq. (2) and present some alternatives. In Section 4, we analyze the consequences of considering different velocity profiles for the rolling phase and discuss the connections with other models for granular materials in the literature. Also, we discuss the role played by some of the parameters in the model. In Section 5, we present a new and simple model to describe the mechanism that underlies the exchange of grains between static and rolling phases. We are then able to deduce an expression similar to Eq. (2). Finally, in the last Section, we summarize our results.

2. Mass conservation

Consider, for instance, a two-dimensional sandpile with a rolling and a static phase. The rolling phase is located above the static phase, and slides over it (see Fig. 1). Let us assume that the sandpile can be treated as a continuous medium, ρ_r being the area density of the rolling phase. The mass inside an interval x_o to $x_o + \delta x$, where δx is small, is then given by

$$\delta m_r = \int_{x_o}^{x_o+\delta x} \left[\int_{h(x,t)}^{h(x,t)+r(x,t)} \rho_r(x,y,t)\,\mathrm{d}y \right] \mathrm{d}x \,, \tag{3}$$

where $h(x,t)$ and $r(x,t)$ are the heights of the static and rolling phases, respectively, at position x and time t.

We now define the linear density of the rolling phase,

$$R(x_o,t) = \lim_{\delta x \to 0} \frac{\delta m_r}{\delta x} = \lim_{\delta x \to 0} \frac{1}{\delta x} \int_{x_o}^{x_o+\delta x} \left[\int_{h(x,t)}^{h(x,t)+r(x,t)} \rho_r(x,y,t)\,\mathrm{d}y \right] \mathrm{d}x \,. \tag{4}$$

Using the identity

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{x}^{x+\varepsilon} f(u)\,\mathrm{d}u = f(x) \,, \tag{5}$$

we can write Eq. (4) as

$$R(x_o,t)=\int_{h(x_o,t)}^{h(x_o,t)+r(x_o,t)}\rho_r(x_o,y,t)\,\mathrm{d}y\,. \tag{6}$$

If ρ_r is independent of the vertical coordinate y, Eq. (6) becomes

$$R(x,t)=\rho_r(x,t)r(x,t)\,. \tag{7}$$

Repeating this procedure for the static phase, we obtain

$$S(x_o,t)\equiv\lim_{\delta x\to 0}\frac{\delta m_s}{\delta x}=\int_{0}^{h(x_o,t)}\rho_s(x_o,y,t)\,\mathrm{d}y\,, \tag{8}$$

and

$$S(x,t)=\rho_s(x,t)h(x,t)\,, \tag{9}$$

where m_s is the mass and ρ_s (which is again supposed to be independent of y) is the area density of the static phase.

Eqs. (7) and (9) define a one-to-one relation from h and r to S and R, respectively, which maps the two-dimensional sandpile into a one-dimensional problem if the densities of the two phases do not change with y. A particular and frequent case that follows in this category is given by $\rho_r=\rho_s=\text{constant}$. The possibility of this mapping is not really unexpected, since, with this property, the involved variables will only depend on the horizontal coordinate x and the time t.

Under the assumption of a continuous model, we can write a mass conservation equation for the rolling phase. For a one-dimensional fluid of density R that flows under a velocity field v (if v does not depend on the y coordinate), we have

$$\frac{\partial R}{\partial t}+\frac{\partial}{\partial x}(vR)=Q\,, \tag{10}$$

where Q represents the sources or sinks of this fluid. The BCRE model allows the exchange between rolling and static grains. Thus, regarding the rolling phase, the static phase acts like a source/sink at every point (the extra mass gained by the rolling phase equals the lost mass of the static phase). Then, we write

$$\frac{\partial R}{\partial t}+\frac{\partial}{\partial x}(vR)=-\frac{\partial S}{\partial t}\,. \tag{11}$$

We see that Eq. (11) is a slightly modified version of Eq. (1), where, instead of the height h, we are working with the density S. The diffusion term (if there is one) will now depend on the specific shape of the velocity field v. In contrast with Eq. (1), we now guarantee mass conservation for all forms of v. If we assume, for instance, a constant velocity field in Eq. (1), as it has already been done in some previous works [6,3,4], the diffusing term has to be discarded in order to ensure mass conservation.

In 1997, Makse [6] applied the BCRE model to a mixture of two grains. He wrote Eq. (1), with constant velocity and without diffusion for each kind of grain,

$$\frac{\partial R_i}{\partial t}+v_i\frac{\partial R_i}{\partial x}=\Gamma_i,\quad i=1,2\,, \tag{12}$$

and simulated this model with $v_1=v_2$, showing that, depending on the value of the parameters, the grains either segregate or stratify. The assumption that $v_1=v_2$ was not justified in that paper. To exemplify the advantages of our equations, applying Eq. (11) to a mixture of two grains, and considering that the mixture is rolling with a constant average velocity v, we can write

$$\frac{\partial(R_1+R_2)}{\partial t}+v\,\frac{\partial}{\partial x}(R_1+R_2)=\Gamma\,, \tag{13}$$

which is similar to Eq. (12), obtained by Makse. However, now it is easy to see that $v_1=v_2$ is not an ad hoc assumption, but rather a requirement of the model.

One more advantage of Eq. (11) is that now we can obtain a whole series of different granular flow regimes by varying the velocity profile of the rolling phase. This point will be discussed in Section 4.

3. Exchange of grains between static and rolling phases

We now make some considerations about the second equation of the BCRE model, that we call exchange equation. In analogy with Eq. (11), we first write an equation similar to Eq. (2) but now in terms of the new variables R and S,

$$\frac{\partial S}{\partial t}=R\left[\gamma\left(\frac{\partial S}{\partial x}+\rho_s\tan\theta_r\right)+\kappa\frac{\partial^2 S}{\partial x^2}\right]\,. \tag{14}$$

Boutreux and Raphaël [5] proposed a modified version of this equation to take into account a shielding effect that is present on the upper grains of the rolling phase due to the lower grains of the same layer. With an adaptation to the variables R and S, this shielding effect can be introduced into Eq. (14) which is then written as

$$\frac{\partial S}{\partial t}=\frac{R\xi'}{R+\xi'}\left[\gamma\left(\frac{\partial S}{\partial x}+\rho_s\tan\theta_r\right)+\kappa\frac{\partial^2 S}{\partial x^2}\right]\,, \tag{15}$$

where ξ' is a small constant related to the thickness of the layer of rolling grains that indeed interact with the static phase. Note that, if $R\sim\xi'$ (a thin rolling phase), $R\xi'/(R+\xi')\sim R$, but if $R\gg\xi'$, then $R\xi'/(R+\xi')\sim\xi'$.

Eq. (15) is still not adequate to describe what happens close to the interface, in the presence of irregularities. As the grains flow, they can erode part of the static phase, creating a (sometimes big) crater with a positive slope in the right border (see Fig. 2). To see this effect, let us analyze a particular case, where the densities are constant.

The term $\partial S/\partial x$ is directly related to the slope of the pile (given by $\partial h/\partial x$). If it is negative, the pile is inclined to the right (and if it is positive the pile is inclined to the left). There is no problem when the slope is negative. As expected, for a local slope above the repose angle, we have erosion (and for a slope below it we have acrescion). If this term is positive, we always have acrescion, which is not a reasonable behavior. To correct this behavior, we suggest a modification, where we consider the minus sign of the modulus of $\partial S/\partial x$,

$$\frac{\partial S}{\partial t}=\frac{R\xi'}{R+\xi'}\left[\gamma\left(\rho_s\tan\theta_r-\left|\frac{\partial S}{\partial x}\right|\right)+\kappa\,\frac{\partial^2 S}{\partial x^2}\right]\,. \tag{16}$$

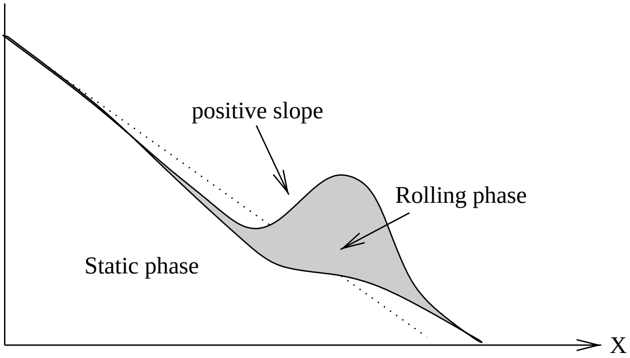

Fig. 2. The flow of rolling grains can eventually erode the static phase creating a crater, and generating regions of positive slope.

4. Diffusion and velocity profiles

One interesting feature of the BCRE model is the possibility of diffusion of the rolling grains. The presence of diffusion in this phase is observed and, in fact, very obvious. This diffusion, however, leads to some constraints on the velocity profile v.

The correct expression for the velocity field v in Eq. (11) should, in principle, be derived from a momentum conservation equation. However, it is not an easy task to write such an equation, since it should take into account all the interactions between the grains and should depend on the stress tensor of the material. In general, we can say that the velocity field is a function of x and t that depends on a variety of factors. For simplicity, most of the works in the literature assume a constant velocity profile.

We now analyze two possible functional forms of v, and report some results of computer simulations to study the effects on the shape of the pile.

In all computer simulations we integrate the BCRE equations numerically by means of a finite difference scheme (see Appendix) and construct an online animation of the profile of both phases in real time. The figures presented below are snapshots of the animation generated by the program. We considered the following profiles:

(1) $v = v(R) = \alpha\partial_x R + \beta R + \delta$, where α, β and δ are constants.

We first take v as a function of R only, and perform a kind of gradient expansion. Note that, in this case, v is an implicit function of x and t ($v = v(R)$ where $R = R(x,t)$). From Eq. (11) we have

$$\frac{\partial R}{\partial t} + \frac{\partial}{\partial x}(vR) = \frac{\partial R}{\partial t} + \alpha\left(\frac{\partial R}{\partial x}\right)^2 + (2\beta R + \delta)\frac{\partial R}{\partial x} + \alpha R\,\frac{\partial^2 R}{\partial x^2}\,. \tag{17}$$

Note that the functional form of v includes a diffusion term. From this last equation, it can also be seen that:

(i) The diffusion coefficient is proportional to R. This means that, the higher the pile, the more it will diffuse. This is reasonable and can be understood as a consequence of gravity.

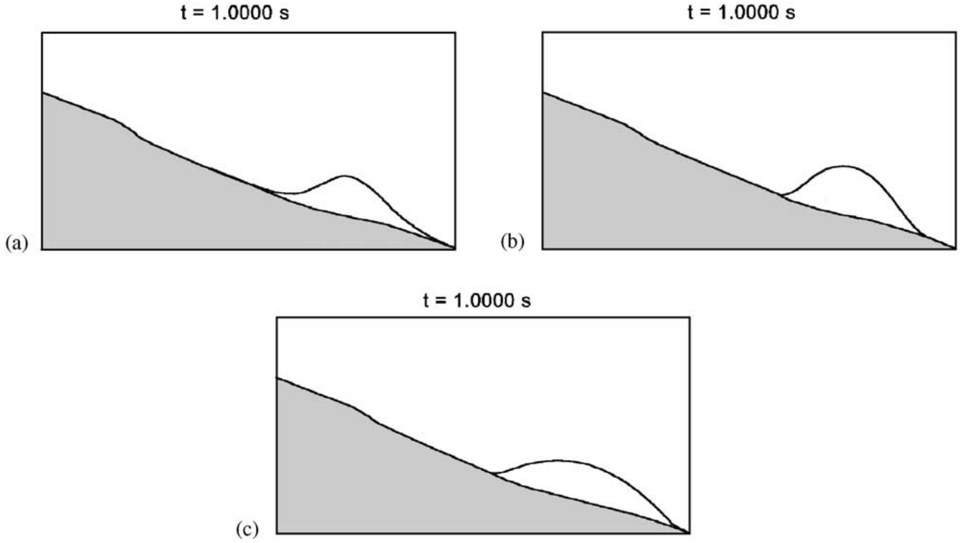

Fig. 3. Profile, at equivalent time t and same initial conditions, of the evolution of a pile in the following cases: (a) the original model, with constant velocity ($v = 1$); (b) model with mass conservation, given by Eq. (11), with $\alpha = -0.01$ and (c) the same of (b) but now with $\alpha = -0.04$. Gray area corresponds to the static phase and white area to the rolling phase.

(ii) There is a non-linear term with a constant coefficient α ($\alpha \neq 0$). If diffusion is small and can be neglected (as in most cases studied in the literature), the non-linear term is unimportant, and the above equation is in agreement with previous works. Diffusion, however, affects the profile of the pile. Fig. 3 shows the effects of this term. It compares the profile of a pile of grains, at the same time and for the same initial condition, for: (a) the original model with constant velocity; (b) Eq. (17), as proposed by us, with $v = \alpha\partial_x R$, for $\alpha = -0.01$; and (c) the same as (b), but with $\alpha = -0.04$. We can see that the final shape of the bump is quite different, if α is not too small.

(iii) This equation has also an advective term, i.e., a term with a first derivative of R with respect to x. Its coefficient depends linearly on R (for $\beta \neq 0$). Note that a similar term has already been proposed in other papers, as in [8]. Fig. 4 shows how it affects the profile of the pile. We can see that there is a tendency to the formation of shock fronts in one of the sides of the bump.

(2) $v = f(\partial_x h)$

In a real pile of grains, there may be irregularities with positive slope, due to erosion. In this case, the velocity must depend on the sign of the slope, otherwise we will have avalanches climbing up the pile at the points with positive slope, with the same velocity as in the negative slope side. To correct this defect, we have taken

$$v = \begin{cases} \alpha\partial_x R + \beta R + \delta, & \text{if } \partial_x h \leqslant 0\,, \\ v_s, & \text{if } \partial_x h > 0\,, \end{cases} \tag{18}$$

where v_s is a constant.

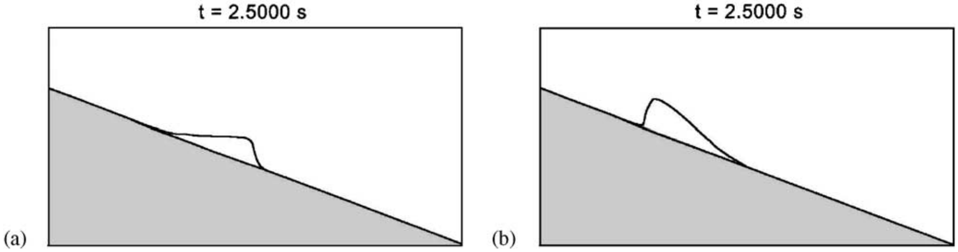

Fig. 4. Effects of diffusion. Profile, at equivalent time t and same initial conditions, of the evolution of a pile with a velocity profile given by $v = \alpha\partial_x R + \beta R + \delta$, with $\delta = 1$. $\alpha = 0.01$ and (a) $\beta = 2$, or (b) $\beta = -1$. A negative β means that the thicker the rolling phase is, the more difficult it is to the grains to roll. Gray area corresponds to the static phase and white area to the rolling phase.

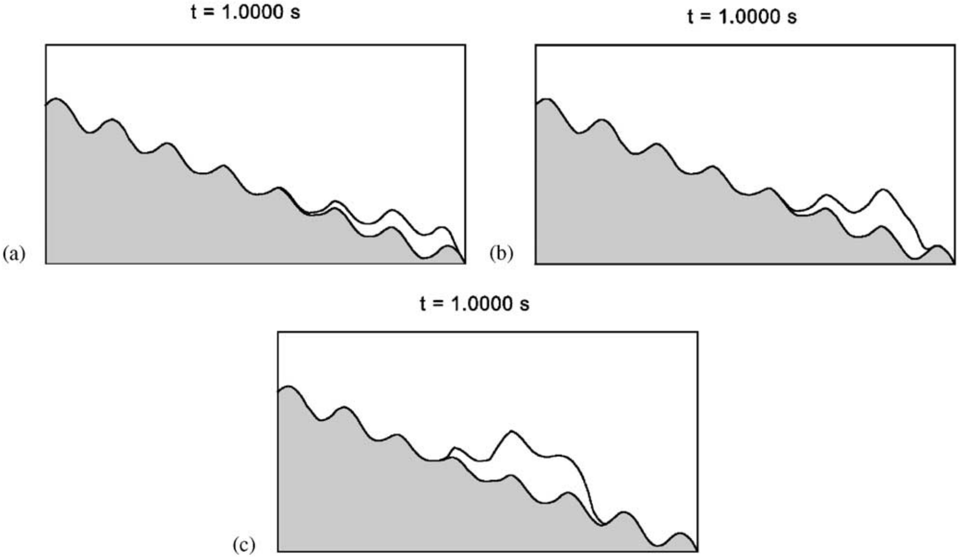

Fig. 5. Profile, at equivalent time t and same initial conditions, of the evolution of a pile in the following cases where (a) the velocity is independent of the sign of slope and (b) the velocity depends on the sign of the slope and $v_s = 0.9$ and (c) the same of (b) but now with $v_s = 0.4$. Gray area corresponds to the static phase and white area to the rolling phase.

For $\partial_x h \leqslant 0$, this expression is equivalent to the previous form of v. But, for $\partial_x h \geqslant 0$, the rolling grains meet a barrier of static grains, and move up this barrier with a constant velocity. There is no diffusion in this case, since the velocity is constant. But now this is a desired property; the grains slowly accumulate in the barrier (and do not diffuse). Fig. 5 shows the profile of the pile at the same time but for different forms of v: (a) if the velocity is independent of the slope; (b) if v depends on the slope according to Eq. (18), for $v_s = 0.9$, and (c) the same as in (b), but with $v_s = 0.4$. The static phase is gray, and has been settled to an irregular shape to amplify the effects of the changes.

At the final stages of this work, we learned about a work of Herrmann and Sauermann on the behavior of the Barchan Dunes [9] that was also based on the BCRE model, and dealt with some of the points we present in this paper. They consider a two-dimensional version of Eq. (11) but they do not deduce it. In particular, there is no mention of the need of independence of the densities with respect to the y coordinate. Indeed, we believe that, although it is not explicitly written, the simulations were performed under constant densities, which is a particular case of y independence. They also used a slightly different version of Eq. (2), with the modulus of the slope of the static phase.

5. Simple model for the exchange of grains between rolling and static phases

On the bases of a naive model, it is possible to deduce an equation to describe the exchange of grains between rolling and static phases. Remember that the densities of the rolling and static phases are different, the latter being larger than the former. As the rolling phase rolls over the static phase, friction removes energy from the rolling grains. Part of this energy becomes heat, but the rest is transferred to the grains of the static phase. This energy agitates grains and the density of the static phase right below the interface decreases, until it reaches the critical dilatancy and starts to move, becoming part of the rolling phase. This process is similar to a solid to liquid first order phase transition, the static phase playing the role of the solid (receives energy and starts to “melt”). Indeed, there is experimental evidence that, at least in the case when the transition is induced by tilting, it does display features of a first order phase transition [10]. We will assume that this analogy is valid. We can then calculate the amount of mass of the static grains that will “melt” (that is, receives energy and starts to roll) using an analogy with the latent heat equation

$$\delta Q = L\delta m_s \,, \tag{19}$$

where δQ is the energy gained from the rolling phase, δm_s is the amount of mass of the static grains that melts, and L is a constant, analogue to the latent heat.

Let us now focus on what happens in a small interval δx of the horizontal coordinate of the pile (see Fig. 1). The amount of mass that is melted is a portion of the total mass of static grains. However, not all the static grains receive energy from the rolling phase. The upper grains shield the lower grains from the contact with the rolling phase. We consider that the amount of mass that can actually receive energy (and, therefore, melt) is a fraction ξh of the static phase (later, we will justify this assumption, that is now adopted for the sake of simplicity).

In the interval δx, the mass that can be melted is given by

$$m_s = \rho_s V = \xi h \delta x \rho_s = \xi h \delta x \frac{S}{h} = \xi S \delta x \,. \tag{20}$$

Thus, the amount of mass that melts is given by $\delta m_s = \delta S \xi \delta x$. Therefore, we have

$$\delta Q = L\xi \delta S \delta x \,, \tag{21}$$

where δS is the change of S due to the melting of the static phase. Note that, if the pile is too high, internal forces and gravity act to de-stabilize it, which increases the melting. This can justify the assumption that m_s is proportional to h (instead of being a constant layer).

If all the energy to melt the static phase comes from the rolling phase, and if it is a fraction of the kinetic energy that is lost due to friction in the interface, we have

$$\delta Q = c\delta K \,, \tag{22}$$

where δK is the kinetic energy lost by the rolling phase. But $K = p^2/2m_r$, where m_r is the mass of the rolling phase and p its momentum. So we have

$$\delta K = \frac{2p\delta p}{2m_r} = \frac{(m_r v)\delta p}{m_r} = v\delta p \,. \tag{23}$$

Recalling that $m_r = R\delta x$, supposing that the rolling phase transfers energy to the static phase only by friction, and that friction is proportional to the weight of the rolling phase at x, we have

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \mu m_r g \Rightarrow \delta p = \mu m_r g \delta t = \mu g R \delta x \delta t \,, \tag{24}$$

From Eqs. (21)–(24), we can write

$$L\xi\delta S\delta x = c\mu g v R\delta t\delta x \,. \tag{25}$$

Thus, in the limit δt, $\delta S \to 0$, we have

$$\frac{\partial S}{\partial t} = \beta v R \,, \tag{26}$$

where $\beta = \frac{c\mu g}{L\xi}$ is a constant.

This is a quite simple expression for the exchange equation. It can assume a variety of forms, depending on the velocity field v. It explicitly incorporates the velocity field, thus indicating that the exchange of grains between both phases depends on the exact shape of v, which is very reasonable, since the velocity of the rolling grains interferes directly with the energy lost in the collisions (which are ultimately responsible for the transformation of the static grains into rolling grains).

Note also that if Eq. (26) is inserted into Eq. (11) the mass conservation equation, we get

$$\frac{\partial R}{\partial t} + v\frac{\partial R}{\partial x} = q \,, \tag{27}$$

where

$$q = \left(\beta v - \frac{\partial v}{\partial x}\right) R \,. \tag{28}$$

If the velocity is a function of x, t and R only, q will also be a function of these three variables (because R is a function of x and t only), and the resulting equation will be a well-known quasi-linear partial differential equation of first order in R, that can be solved by the method of characteristics, given by the simple system (see, for example, Ref. [11])

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{v(x,t,R)} = \frac{\mathrm{d}R}{q(x,t,R)} \,. \tag{29}$$

The only difficulty is that, if v has an explicit dependence on R, the variable q will have a dependence on $\partial R/\partial x$, which may turn the system of characteristics difficult to be solved analytically. However, a solution for $R(x,t)$ will consequently give a solution for $S(x,t)$ by means of Eq. (26). We intend to further explore this point in a following paper.

Furthermore, if Eqs. (26) and (16) are equivalent, the velocity field must assume the form:

$$v = \frac{\xi'/\beta}{R+\xi'}\left[\gamma\left(\rho_s \tan\theta_r - \left|\frac{\partial S}{\partial x}\right|\right) + \kappa\frac{\partial^2 S}{\partial x^2}\right] . \tag{30}$$

This velocity field has some interesting features. First, note that v depends on the slope of the pile. The first term inside the square brackets becomes positive for a slope below the angle of repose and negative above it, which means that the velocity is higher as the pile is steeper. Second, the factor $(R+\xi')^{-1}$ suggests that v is inversely proportional to the weight of the pile of rolling grains.

6. Conclusions

In the first two sections of this paper, we pointed out some problems with the equations of the BCRE model, suggesting that they should be changed and written as:

$$\frac{\partial R}{\partial t} + \frac{\partial}{\partial x}(vR) = -\frac{\partial S}{\partial t} , \tag{31}$$

and

$$\frac{\partial S}{\partial t} = \frac{R\xi'}{R+\xi'}\left[\gamma\left(\rho_s \tan\theta_r - \left|\frac{\partial S}{\partial x}\right|\right) + \kappa\frac{\partial^2 S}{\partial x^2}\right] . \tag{32}$$

where $R = \rho_r r$ and $S = \rho_s h$ are the linear densities of the rolling and static phases, respectively, θ_r is the repose angle, v is the velocity profile, γ and κ are positive parameters and ξ' is a small constant.

The first equation explicitly assures mass conservation. The second equation was changed to take into account the sign of the slope in the static phase. In addition, we introduced the new variables R and S, giving a precise definition for them, which was not entirely clear in the literature.

We used Eqs. (31) and (32) to simulate two possible velocity fields, and found acceptable results. Also, we have proposed a model for the exchange of grains between the rolling and static phases. We then obtained the alternative $\partial S/\partial t = \beta v R$ for the exchange equation, that is simpler, includes the velocity field explicitly, and leads to interesting results.

We are aware that many of the underlying hypothesis of this simple model must be examined in more detail. We assumed that friction is proportional to the weight of the rolling phase, and this is surely oversimplified. Probably, there is a more complicated dependence on other parameters of the model as well. For instance, it is reasonable to suppose that the energy transferred to the static phase also depends on the shape of the grains, its density, the toughness of the material and a variety of other factors.

We believe that the fraction of static grains that receives energy from the rolling phase is probably a more general function of h, not to mention explicitly a dependence on the other variables of the model.

We have also neglected the inverse process, i.e., the transformation of rolling grains into static ones. Our model may be good to describe an avalanching process, where the inertia of the rolling grains is large, but may fail to describe a more general situation.

We hope to address these points in a following paper. However, we think that it is already very interesting that a somewhat richer expression for v, as Eq. (30), can be obtained from such a naive model.

Acknowledgements

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Appendix A

The equations of the BCRE model were integrated with the operator splitting method [12]. Suppose a differential equation of the form:

$$\frac{\partial u}{\partial t} = L(u)\,,$$

where $L(u) = \sum_{i=1}^{m} L_i(u)$ is a generic non-linear operator that can be written as a sum of m other operators, and u is a function of x and t. If we have a good method to integrate each of the equations $\partial u/\partial t = L_i(u)$, then u^{n+1} can be obtained through m successive time steps,

$$\begin{aligned} u^{n+(1/m)} &= L_1(u^n, \delta t)\,, \\ u^{n+(2/m)} &= L_2(u^{n+(1/m)}, \delta t) \\ \vdots &= \vdots \\ u^{n+1} &= L_m(u^{n+(m-1)/m}, \delta t)\,. \end{aligned}$$

For Eq. (1) (and its variants), the L_i operators are of the form:

$$L_1(r) = f(r)\frac{\partial r}{\partial x}, \quad L_2(r) = g(r)\frac{\partial^2 r}{\partial x^2}, \quad L_3(r) = q(r)\,,$$

and

$$L_4(r) = k\left(\frac{\partial r}{\partial x}\right)^2 ,$$

where f, g and q are arbitrary functions of r, and k is a constant. L_1 and L_2 were integrated with variants of the Crank–Nicholson method; the operator L_3 was integrated with a fourth order Runge–Kutta procedure, and L_4 was integrated with a FTCS finite-difference scheme.

The second equation can be split into two operators of the form:

$$L_1(h) = a\,\frac{\partial^2 h}{\partial x^2}\,,$$

and

$$L_2(h) = b\,\frac{\partial h}{\partial x}\,,$$

where a and b are constants. Now L_1 was integrated with the aid of Crank–Nicholson and the operator L_2 with the aid of a two-step Lax–Wendroff [12] procedure.

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