Network Information Flow for multicast communication networks

Project report submitted in partial fulfillment

of the requirements for the degree of

MASTER OF TECHNOLOGY

in

COMPUTER SCIENCE AND ENGINEERING

Submitted by

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M-Tech III Year

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December 06, 2006

Certificate

This is to certify that the project report titled "**Network Information Flow for multicast communication networks**" being submitted by Nikhil Bhargava (2004MCS2650) for the partial fulfillment of the requirements for the degree of Master of Technology is a bonafide work carried out by him under my guidance and supervision. He has also been co-guided by professor S.N Maheshwari at *Computer Science and Engineering Department, Indian Institute of Technology, Delhi.* The work presented in this report has not been submitted elsewhere, either in part or full, for the award of any degree.

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Acknowledgement

I would like to thank the following persons help me complete my research work on time: Dr. Naresh Sharma, my supervisor, for his encouragement, patience and expert advice; Dr. S.N Maheshwari, my co-advisor for his guidance and time and finally special thanks go to my friends and family members who have supported me throughout this work.

Nikhil Bhargava

Abstract

Increasing throughput and maximizing bandwidth usage is a potential requirement in both unicast and multicast comunication networks. For any given point-to-point communication network with multiple mutually exclusive information sources mulitcasting information bits to some sets of destinations, it is difficult to characterize the admissible coding rate region. Network switching alone can never reach to max-flow min-cut bound however, by employing coding at the nodes, referred to as network coding, bandwidth can in general be saved. Ahlswede, Cai, Li, and Yeung have shown that maximum achievable information rate by network coding in a single source multicast network is more than that for the case of network switching. WE have this problem of finding the maximum admissible coding rate region for a given network and then devising a suitable information broadcast strategy for a given communication network. We have analyzed the switching gap for a special network, defined as the ratio of maximum achievable information rate using network coding (NC) to that of network switching (NS). In this work, I have found the switching gap of a singular-symmetric, dual-symmetric, triple-symmetric and then give an intuitive observation for the n^{th} version of singular symmetric butterfly network which I term as generic butterfly network.

Keywords: Network switching, Network coding, max-flow min-cut theorem, multicast networks, convex Optimizations, Linear Programming, Route packing

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Chapter 1

Introduction

In their pioneering work on network coding [1], Ahslwede et al determined the capacity for multicasting information in a network of lossless channels. Specifically, the network is represented as $G = (V,E,c)$, where (V,E) is a directed graph and c is the edge capacity vector of length $|E|$. Subsequently, Li, Yeung, and Cai [2] showed that the multicast capacity can be achieved by linear network coding.

Ahslwede et al. in [1] have established a theory of network coding for single-source and multisource-source. For single-source network coding, they demonstrated that the maximum achievable information rate for network coding is always upper bounded by max-flow min-cut bound. Further, they demonstrated by many examples that this bound cannot be achieved by conventional network switching and some kind of network coding [3] has to be applied at each node of the network. Since past 5 years, recent discoveries in this field have generated a lot of interest in research fraternity. A comprehensive survey on the theory of network coding is presented in Yeung, Li, Cai, and Zhang in [4].

Network switching and network coding are two techniques of data combination to increase throughput although the former is a special case of the latter. Switching gap [5], is defined as the ratio of maximum achievable information rate using network coding (NC) to that of network switching (NS). A natural problem arises from here, what is the switching gap for a given single source multicast network and under what conditions does switching gap reaches its maxima and minima (which will be 1). Xue-Bin Liang in [6] has analyzed the Ahlswede-Cai-Li-Yeung's classical butterfly network [1] to determine switching gap of the network as well as determined certain conditions on link capacities such that switching gap of the network comes out to be unity.

In first part of major project, I have looked on the problem of network information flow in a multicast network. Post[1] era, many people have worked on developing optimal strategies for network information flow for different class of networks. I have tried to extend Liang's work [1] by taking modified versions of butterfly network and finding its switching gap under a given set of conditions on the link capacities of the network. In this process, I analyze singular-symmetric, dual-symmetric and triple-symmetric butterfly network and then give an expression for a special case of a generic butterfly network. Apart from this, I am trying to work out network switching gain for a class of network by devising an optimal switching strategies for all min cuts for the network.

This chapter gives a short introduction to the problem. The rest of the report is organized as follows. Next chapter gives motivation for this problem and later gives formal problem statement. Chapter 3 gives an introduction to classical max flow min-cut theorem and its applications. Chapter 4 gives a detailed note on game theory and dominance relationships used in next chapter to calculate the network switching gain. Chapter 5 gives the gives analysis of singular-symmetric butterfly network, dual-symmetric butterfly network, triple-symmetric butterfly network and finally generic butterfly network. Chapter 6 concludes the report and highlights course for next part of the project.

Chapter 2

Motivation and Problem Statement

A lot of research in the realm of network coding have happened in recent years which provides the motivation for our work. Let $G = (V, E)$ a point-to-point communication network represented by a directed graph, where V is the finite set of vertices in the network and E is the finite set of edges connecting two vertices say u and $v \in V$ in the network G. We do not consider any edge connecting a node to itself i.e., in other words we are not considering multigraph. Each edge or link, say e is associated with a capacity, c which belongs to set of positive rational numbers, R^+. This network could be used to transmit information from one node to other in the network. We can safely assume that transmission in the network will be error free if and only if, transmission rate, r over any link e should be lesser then link capacity, c. Let X be the information source, generating information in bit sequences spread over the field Ψ node $s \in V$ in the network G. Information gets transmitted from s to every destination or sink nodes, t_1, $t_2, \ldots$, $t_L \in V$ such that information could be reconstructed at each of t_i. Hence, without loss of any generality, we can say that information from source X gets multicast to L distinct sinks in G. This boils down to a single source-multisink problem in a multicast network. Please note that for both Network switching (NS) and Network coding (NC), there will not be any information loss, however flow might not be conserved.

An interesting problem which is only partially explored is that, when is information rate achieved by network coding (NC) equal to information rate achieved by (NS) under a given set of conditions on link capacities. For a generic network showing no symmetry, it would be a tough problem to solve. We have tried to solve this problem for a specific case that could be considered as a generic version of Ahlswede-Cai-Li-Yeung's butterfly network after modifying this network by taking capacities of all links at same level to be equal (see section chapter 5 for further details).

The next chapter gives a short introduction of network flows and Max-Flow Min-Cut theorem.

Chapter 3

Max-Flow Min-Cut Theorem

In graph theory, a network flow is an assignment of flow to the edges of a directed graph, called a flow network in this case, where each edge has capacity (which may be positive real or integer), such that the amount of flow along an edge does not exceed its capacity. Further, there is restriction that the amount of flow into a node equals the amount of flow out of it, except if it is a source, which only has outgoing flow, or sink, which has only incoming flow. A flow network can be used to simulate traffic in a road system, fluids in pipes, currents in an electrical circuit, or anything similar in which something travels through a network of nodes.

Given a graph G (V,E) with nodes V and edges E, and special nodes source s (in-degree 0) and sink t (out-degree 0). Let $f(u,v)$ be the flow from node u to node v, and $c(u,v)$ the capacity. Formally stating, a network flow is a real function f: $V \times V \rightarrow R$ with the following three properties for all nodes u and v:

1. Skew Symmetry: $f(u,v)$ = -$f(v,u)$
2. Capacity Constraints: $f(u,v) \leq c(u,v)$
3. Flow conservation: $\sum_{w \in V} f(u,w) = 0$, where $w \notin (s,\ t)$

The residual capacity of an edge is $c_f(u,v) = c(u,v)$ - $f(u,v)$. A residual network is denoted by G_f (V,E_f) and it consists of residual edges obtained by transforming the original network G. This way there can be an edge from v to u in the residual network, even though there is no edge from u to v in the original network.

A cut (S,T) of flow network G=(V,E) is a partition of V into S and T = V-S such that $s \in S$ and $t \in T$.

An augmenting path is a path (u_1, u_2, ..., u_k), where $u_1 = s$, $u_k = t$, and c_f (u_i, $u_i + 1$) > 0, such that more flow could be pushed along this path. There are various ways of choosing an augmenting path and depending upon that different max flow determination algorithms have been proposed.

3.1 The Maximum-Flow Min-Cut Theorem

Theorem : If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the 3 following statements are equivalent

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.

Chapter 4

Game Theory

Game theory is a branch of mathematical analysis developed to study decision making in conflict situations. Such a situation exists when two or more decision makers who have different objectives, act on the same system or share the same resources. There are two person and multi person games. Game theory provides a mathematical process for selecting an *Optimum strategy* (that is, an optimum decision or a sequence of decisions) in the face of an opponent who has a strategy of his own. In game theory one usually makes the following assumptions:

1. Each decision maker *player* has available to him two or more well-specified choices or sequences of choices called *plays*.

2. Every possible combination of plays available to the players leads to a well-defined end-state (win, loss, or draw) that terminates the game.

3. A specified payoff for each player is associated with each end-state (a *zero-sum* game means that the sum of payoffs to all players is zero in each end-state).

4. Each decision maker has perfect knowledge of the game and of his opposition; that is, he knows in full detail the rules of the game as well as the payoffs of all other players.

5. All decision makers are rational; that is, each player, given two alternatives, will select the one that yields him the greater payoff.

Game Theory finds wide application in areas of financial accounting, economics, sociology, computer networks, stochastic based applications etc. Network switching can be formulated as a special game, called matrix game played in the multicast networks [7, 12]. Interested readers can refer [9, 10, 13] for further details.

4.1 Zero-Sum games

Formally, a game τ is said to be zero-sum if and only if at each terminal vertex of the game tree, the payoff function $(p_1,...,p_n)$ satisfies

$$\sum_{i=1}^{n} p_i = 0.$$

Figure 4.1 shows Game Tree for matching pennies. It is a very simple example of zero-sum with two players. It consists of tossing a coin by each player and in case both outcomes are heads or both are tails then player One wins, otherwise Two wins.

A finite zero sum 2 person game reduces to a matrix A, with as many rows as Player PI has strategies and as many columns as player PII has strategies. In simpler terms, the payoff is defined as the amount first Player PI receives from second Player PII. PI will try to maximize it while PII will try to minimize it. It is a play of game such that, if PI chooses say, i^{th} row from the I rows and PII chooses say, j^{th} column from the J columns, then the expected payoff, is the element a_{ij}, in the i^{th} row and j^{th} column of the matrix.

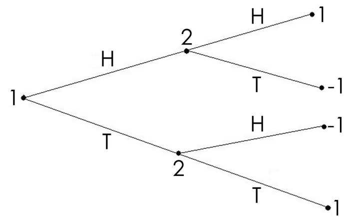

Figure 4.1: Game of matching Pennies

We refer to rows and columns as pure strategies of PI and PII respectively. Aim of PI is to maximize the minimum payoff, thereby guaranteing a lower bound called (*gain floor*) given by

$$\overline{\nu} = \max_{1 \le i \le I} \min_{1 \le j \le J} a_{ij}$$

Similarly, PII will try to choose a pure strategy so as to minimize the maximum payoff thereby, guaranteing an upper bound for loss (called *loss ceiling*) given by

$$\underline{\nu} = \min_{1 \le j \le J} \max_{1 \le i \le I} a_{ij}$$

It is clear that $\underline{\nu} \geq \overline{\nu}$ and equality holds $\Leftrightarrow$ there exists a pair of strategies (i^*, j^*) satisfying the following condition.

$$\min_{1 \le j \le J} a_{i^* j} = a_{i^* j^*} = \max_{1 \le i \le I} a_{ij^*} \tag{4.1}$$

(i^*,j^*) satisfying (4.1) is called a *saddle* point for the payoff matrix A. It is a also termed as Nash equilibrium point of the game since, at this point both the players are having maximum gain and one way change of strategy by either player will not provide any gain. Saddle point may or may not exist. For e.g., consider this game matrix.

$$\begin{vmatrix} 5 & 1 & 3 \\ 3 & 2 & 4 \\ 3 & 0 & 1 \end{vmatrix}$$

It has a saddle point at 1^{st} row 2^{nd} column with ν=1.

4.2 Mixed Strategy

A mixed strategy for a player is a probability distribution on the set of his pure strategies. Precisely, if say 'm' pure strategies are there, a mixed strategy reduces to m-vector, x=(x_1, ..., x_m), satisfying

$$x_i \geq 0 \quad \text{and}$$

$$\sum_{i=1}^{m} x_i = 1.$$

Let X be the set of all mixed strategies of PI and let Y be the set of all mixed strategies of PII. If PI chooses mixed strategy x and PII chooses y, then the expected payoff is

$$\begin{aligned} A(x,y) &= \sum_{i=1}^{m}\sum_{j=1}^{m} x_i a_{ij} y_i \text{ or in matrix notation} \\ A(x,y) &= xAy^T \\ \nu_{\mathrm{I}} &= \text{PI's gain floor} \\ &= \max_{x\in X}\min_{j}\ xA\cdot j \\ &\quad (A\cdot j \text{ is the } j^{th} \text{ column of } A) \\ \nu_{\mathrm{II}} &= \text{PII's loss ceiling} \\ &= \min_{y\in Y}\max_{i}\ A\cdot iy^T \end{aligned}$$

We can represent mixed strategy for PI by an I-dimensional probability distribution X = $(x_1,\ x_2,\ ...,\ x_I)^T$, where T denotes the transpose of a matrix. A mixed strategy for PII is denoted by a J-dimensional probability distribution vector Y=$(y_1,\ y_2,\ ...,\ y_J)^T$. So, the *expected payoff* in case PI chooses mixed strategy x and PII chooses mixed strategy y is

$$x^T A Y = \sum_{i=1}^{I}\sum_{j=1}^{J} x_i a_{ij} y_j.$$

Let us define X as

$$\begin{aligned} \mathrm{X} &= \{(x_1, x_2, ..., x_I) \in R^I \mid \\ &\quad \sum_{i=1}^{I} x_i = 1 \text{ and } x_i \geq 0 \text{ for } i = 1, 2....I\}. \end{aligned} \tag{4.2}$$

Now we can define PI's *expected gain-floor* (ν_{I}) and PII's *expected loss ceiling* (ν_{II}).

$$\nu_{\text{I}} = \max_{x\in X}\min_{y\in Y} x^T A Y. \tag{4.3}$$

$$\nu_{\text{II}} = \min_{y\in Y}\max_{x\in X} x^T A Y. \tag{4.4}$$

MiniMax Theorem : It states that

$$\nu_{\text{I}} = \nu_{\text{II}} \tag{4.5}$$

Minimax Theorem implies that every matrix game with payoff matrix as A has at least one pair of mixed strategies (x^*,y^*) where in, $x^* \in X$ and $y^* \in Y$ such that

$$\nu_{\text{I}} = \min_{y\in Y}(x^*)^T Ay = (x^*)^T Ay^* = \max_{x\in X} x^T Ay^* = \nu_{\text{II}}. \tag{4.6}$$

So (x^*,y^*), is the saddle point of the expected payoff function $x^T Ay$. Following Lemma [13, pg. 138, Eqn. (5)] characterizes the solution of a matrix game.

Lemma : For a matrix game with $I\times J$ payoff matrix A, a necessary and sufficient condition for a mixed-strategy pair (x^*, y^*) given $x^* \in X$ and $y^* \in Y$ to be a Nash Equilibrium point and for a real number $\nu \in R$ to be the value of the game is that every component of the vector $(x^*)^T A \in R^J$ is $\geq \nu$ and every component of the vector $Ay^* \in R^I$ is $\leq \nu$.

4.3 Computation of Optimal Strategies

In case,if a saddle point exists, then pure strategies i and j or equivalently, the mixed strategy x and y with x_i=1, y_i=1 and all other components equal to zero, will be optimal strategies of PI and PII respectively.

Domination : In a matrix A, we say that the i^{th} row dominates the k^{th} row if

$$\begin{aligned} a_{ij} &\geq a_{kj} \text{ for every } j \text{ and} \\ a_{ij} &> a_{kj} \text{ for atleast one } j \end{aligned}$$

Similarly, we say that j_{th} column dominates the i_{th} column if,

$$\begin{aligned} a_{ij} &\leq a_{il} \text{ for every } i \text{ and} \\ a_{ij} &< a_{il} \text{ for at least one } i \end{aligned}$$

Here is a working example

$$A = \begin{vmatrix} 2 & 0 & 1 & 4 \\ 1 & 2 & 5 & 3 \\ 4 & 1 & 3 & 2 \end{vmatrix}$$

2^{nd} column dominates the 4^{th} column $\Rightarrow$ PII will never use 4^{th} column.

$$A = \begin{vmatrix} 2 & 0 & 1 & \underline{4} \\ 1 & 2 & 5 & \underline{3} \\ 4 & 1 & 3 & \underline{2} \end{vmatrix}$$

3^{rd} row dominates 1^{st} row $\Rightarrow$ PI will never use 1^{st} row

$$A = \begin{vmatrix} \underline{2} & \underline{0} & \underline{1} & \underline{4} \\ 1 & 2 & 5 & \underline{3} \\ 4 & 1 & 3 & \underline{2} \end{vmatrix}$$

3^{rd} column dominates first column. So, eliminating all dominated components, we have

$$A = \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$$

So we need solve just the 2×2 game-matrix.

Another way to solve for optimal strategy for matrix games is by fictious play method given in [10] but it works for integral link capacities. Moreover, it is intuitive and lacks formal basis.

Chapter 5

Results

This chapter gives the results of the work done till date. I have constructed different variations of classical butterfly network and calculated the switching gap for each of them. Based upon the results, I have made an intuitive observation about generic version of this butterfly network.

5.1 Analysis for Singular Butterfly Network

Figure 5.1 shows singular butterfly network. Here w_i >0 denotes the link capacities of the edges in the graph.

Now, according to Ahlswede-Cai-Li-Yeung's fundamental theorem for single-source network coding [1], the maximum achievable information rate denoted by R^{**} is equal to the minimum of the s-t cuts for all source-sink pairs in the network. To enumerate the s-t cuts, consider the subgraph $G' = (V', E')$, where $V' \subset V$ and $E' \subset E$, which is formed from graph shown in above figure by removing all paths between s-t_2. All s-t cuts are shown as dashed lines in figure 5.2. The cut-set for this network is enumerated below.

1. $\{(s,a),(s,b)\} = 2w_1$

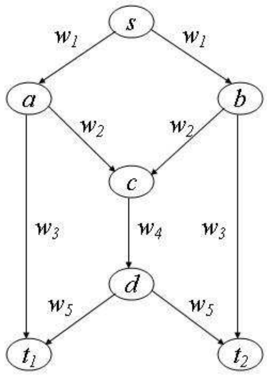

Figure 5.1: Singular-Symmetric Butterfly network

2. $\{(s,a),(b,c)\} = w_1 + w_2$

3. $\{(s,a),(a,c),(d,t_1)\} = 2w_2 + w_3$

4. $\{(s,a),(a,c),(c,d)\} = w_1 + w_2 + w_4$

5. $\{(a,t_1),(a,c),(b,c)\} = w_1 + w_2 + w_5$

6. $\{(a,t_1),(c,d)\} = w_3 + w_4$

7. $\{(a,t_1),(d,t_1)\} = w_3 + w_5$

Similarly, for subgraph between s and t_2, we have the following s-t cut sets.

1. $\{(s,a),(s,b)\}$=$2w_1$

2. $\{(s,b),(a,c)\}$=$w_1 + w_2$

3. $\{(b,t_2),(a,c),(b,c)\}$=$2w_2 + w_3$

4. $\{(s,b),(b,c),(c,d)\}$=$w_1 + w_2 + w_4$

5. $\{(s,b),(b,c),(d,t_2)\}$=$w_1 + w_2 + w_5$

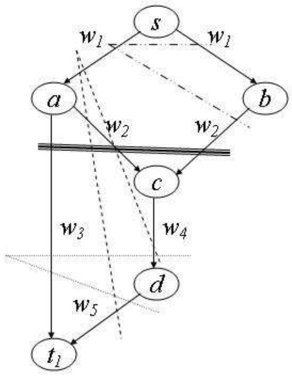

Figure 5.2: Singular-Symmetric Butterfly sub network

6. $\{(b,t_2),(c,d)\}$=$w_3 + w_4$

7. $\{(b,t_2),(d,t_2)\}$=$w_3 + w_5$

For the given singular butterfly network, we have assumed the following conditions

$w_1 < w_2$

$w_1 < w_3$

$w_4 < w_5$

Based upon these three assumptions, we combine the various $s-t$ cut set values and finally, R^{**} is equal the minimum of the following 5 values.

1. $\min(w_1,w_3)+\min(w_4,w_5)$

2. $w_1+\min(w_1,w_2)$

3. $2w_2+w_3$

4. $w_1+w_2+w_3$

5. $w_1+w_2+w_3+w_4$

Under these conditions, we have R^{**} is minimum of

$$\begin{aligned} R^{**} &= \min(w_1 + w_4, 2w_1). \\ R^{**} &= w_1 + \min(w_1, w_4). \end{aligned} \tag{5.1}$$

Now, we are left with problem to solve R^* i.e. maximum achievable information rate by network switching (NS). [7] gives a method to compute R^* of a network by constructing its payoff-matrix and then solving it as per game theory principles [8] and finally taking reciprocal of it to get R^*. In a nutshell, first we determine multicast routes from source to each of the sink nodes. This is a path enumeration problem and can be done by constructing rooted trees for each of the link and then concatenating one path each from all such trees. Formally, a rooted tree is defined as an acyclic digraph with a unique node, called its root node, which has the property that there exists a unique path from the root node to each other node. We call the set of links, τ as a multicast route of the underlying digraph from the source node s to the sink nodes t_1, t_2, ..., t_l, if the digraph (S, τ) induced by τ is a rooted tree of G with t_1, t_2, ..., t_l, whose root node is s and whose leaves are all sink nodes. If L=1, a multicast route τ is the set of links of an open path from the source node s to the sink node t_1. Then for each multicast route τ_j they have defined an *indicator function* over E as $\chi\tau_j(e_i)$=1 if $e_i \in \tau_j$ and 0 otherwise, for $i = 1, 2, \ldots, I$ and $j = 1, 2, \ldots, J$. They have then created an $I \times J$ payoff matrix A such that

$$a_{ij} = \frac{1}{\Theta(e_i)} \chi\tau_j(e_i)$$

for $i = 1, 2, \ldots, I$ and $j = 1, 2, \ldots, J$ with $\{e_1, e_2, \ldots, e_I\}$ being the set of links E and $\{\tau_1, \tau_2, \ldots, \tau_J\}$ being the set of multicast routes from s to $t_1, t_2, \ldots, t_l$.

Brute force method is to enumerate all multicast routes in the network, form a payoff matrix say A using all links and multicast routes enumerated above and solve it to get its value i.e., val(A). Then $R^* = 1/\text{val}(A)$. For this network given in Fig 5.1, payoff matrix is as follows

$$A = \begin{vmatrix} 1/w_1 & 1/w_1 & 1/w_1 & 1/w_1 & 1/w_1 & 1/w_1 & 0 \\ 1/w_1 & 0 & 1/w_1 & 1/w_1 & 0 & 1/w_1 & 1/w_1 \\ 1/w_3 & 1/w_3 & 1/w_3 & 0 & 0 & 0 & 0 \\ 0 & 1/w_2 & 0 & 1/w_2 & 1/w_2 & 0 & 0 \\ 0 & 0 & 1/w_2 & 0 & 0 & 1/w_2 & 1/w_2 \\ 1/w_3 & 0 & 0 & 1/w_3 & 0 & 1/w_3 & 0 \\ 0 & 1/w_4 & 1/w_4 & 1/w_4 & 1/w_4 & 1/w_4 & 1/w_4 \\ 0 & 0 & 0 & 1/w_5 & 1/w_5 & 1/w_5 & 1/w_5 \\ 0 & 1/w_5 & 1/w_5 & 0 & 1/w_5 & 0 & 1/w_5 \end{vmatrix}$$

However, this is not a feasible method for more complex networks since it will have many more links and multicast routes than the network shown in fig 5.1.

We do not necessarily need an enumeration of all the multicast routes to find R^* [7]. We need to work only on the dominating links and dominating multicast routes. A link is dominated by another link if every multicast routes including the former also includes the latter. A multicast route is dominated by another multicast route if each dominating link in the latter is also in the former. Applying successive elimination method from previous section, we can use dominated links and dominated multicast routes to get a simpler square pay off matrix, then find the value of the game and use it to get R^*.

Using assumptions about link capacities stated earlier to compute R^{**}, we have both the links with capacities w_1 and the single link with capacity w_4 as dominating and these 3 links form 3 multi cast routes. As a result, we get the following payoff matrix.

$$A_1 = \begin{vmatrix} 1/w_1 & 1/w_1 & 0 \\ 1/w_1 & 0 & 1/w_1 \\ 0 & 1/w_4 & 1/w_4 \end{vmatrix}$$

Alternatively, using assumptions about various link capacities stated earlier, we see that row number 1 dominates row number 3 and 4 and 5; row number 1 dominates row number 5 and 6 and row number 1 dominates row number 7 and 8 and 9. Removing all dominated rows we have, the following matrix.

$$\begin{vmatrix} 1/w_1 & 1/w_1 & 1/w_1 & 1/w_1 & 1/w_1 & 1/w_1 & 0 \\ 1/w_1 & 0 & 1/w_1 & 1/w_1 & 0 & 1/w_1 & 1/w_1 \\ 0 & 1/w_4 & 1/w_4 & 1/w_4 & 1/w_4 & 1/w_4 & 1/w_4 \end{vmatrix}$$

Now we see that column number 7 dominates column number 3, 4 and 6. Eliminating column number 3, 4 and 6, we have

$$\begin{vmatrix} 1/w_1 & 1/w_1 & 1/w_1 & 0 \\ 1/w_1 & 0 & 0 & 1/w_1 \\ 0 & 1/w_4 & 1/w_4 & 1/w_4 \end{vmatrix}$$

Now, column 2 dominates column 3 or vice-versa since both are equal. Hence eliminating column number 2, we get

$$A_1 = \begin{vmatrix} 1/w_1 & 1/w_1 & 0 \\ 1/w_1 & 0 & 1/w_1 \\ 0 & 1/w_4 & 1/w_4 \end{vmatrix}$$

Please note that the rows of this matrix shows the dominant links in the singular butterfly network where as the columns denote multicast routes. It is clear from [7] that val(A) = val(A_1). Using Lemma given in section III and applying lagrange multipliers [11] to A, we have following cases.

1. *Case 1:* if $2w_1 \geq w_4$ then we have X^*=($w_1/(2w_1+w_4)$,$w_1/(2w_1+w_4)$,$w_4/(2w_1+w_4)$) and Y^*=(($2w_1$-w_4)/($2w_1+w_4$),$w_4/(2w_1+w_4)$,$w_4/(2w_1+w_4)$) are optimal strategies for PI and PII and val(A) is
$$\text{val}(A) = \frac{2}{(2w_1 + w_4)}$$

2. *Case 2:* if $2w_1 < w_4$ then we have X^*=(1/2,1/2,0) and Y^*=(0,1/2,1/2) and val(A) is
$$\text{val}(A) = \frac{1}{2w_1}.$$

From these two cases, we get

$$\text{val}(A) = \min\left((2/(2w_1 + w_4)), (1/2w_1)\right).$$

Hence

$$R^* = \frac{1}{\text{val}(A)} = \frac{2w_1 + \min(2w_1, w_4)}{2}. \tag{5.2}$$

Using this,

$$R^{**} = w_i + w_j. \tag{5.3}$$

And,

$$R^* = \frac{2}{w_i + w_j + \min(w_i + w_j), (w_k)}. \tag{5.4}$$

Using (5.1) and (5.4), we have

$$\frac{R^{**}}{R^*} = \frac{2(w_1 + \min(w_1, w_4))}{2w_1 + \min(2w_1, w_4)}.$$

Now conditioning on values of w_1 and w_4, we have following four cases.

1. case 1: If $2w_1 < w_4$, then we have

$$\frac{R^{**}}{R^*} = \frac{4w_1}{4w_1} = 1$$

2. case 2: If $w_1 < w_4 < 2w_1$, then we have

$$\begin{aligned} \frac{R^{**}}{R^*} &= \frac{2(w_1 + w_4)}{2w_1 + w_4} \\ \frac{R^{**}}{R^*} &= \frac{2}{1 + \frac{w_4}{2w_1}} \end{aligned}$$

3. case 3: If $w_4 < w_1$, then we have

$$\frac{R^{**}}{R^{*}} = \frac{2(w_1 + w_4)}{2w_1 + w_4}$$
$$\frac{R^{**}}{R^{*}} = \frac{2}{1 + \frac{w_1}{w_1+w_4}}$$

If we have $w_1 = w_4 = w$, then the switching gap becomes,

$$\frac{R^{**}}{R^{*}} = \frac{2(w + w)}{2w + w} = \frac{4}{3} \tag{5.5}$$

5.2 Analysis for dual butterfly network

Figure 2 shows dual butterfly network. Here $w_i > 0$ denotes the link capacities of the edges in the graph.

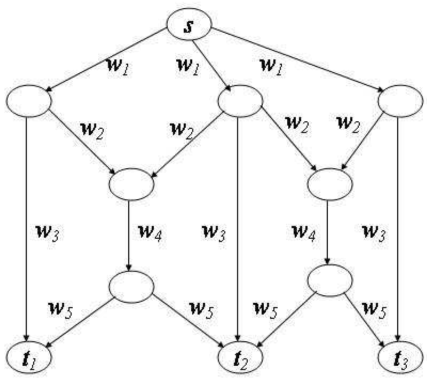

Figure 5.3: Dual-Symmetric Butterfly network

For the above network taking into account assumptions as in case of singular butterfly network, R^{**} is equal the following 7 values.

1. $3w_1$
2. w_1+w_2
3. $w_2+w_3+\min(w_1,w_2)$
4. w_1+w_4
5. $2w_4+w_3$
6. $2w_1+w_3+w_5$
7. $w_1+w_2+w_3+w_4$

From this set under the assumptions in section V, we only take minimum min-cuts and thus

$$R^{**} = \min(3w_1, w_1 + w_2, w_1 + w_4). \tag{5.6}$$

Using assumptions about link capacities stated earlier to compute R^{**}, we have all three links with capacities w_1 and all two links with capacity w_4 as dominating and these 5 links form 5 multi cast routes. After elimination of dominated links and multicast routes, we get the following payoff matrix.

$$A = \begin{vmatrix} 1/w_1 & 1/w_1 & 1/w_1 & 1/w_1 & 0 \\ 1/w_1 & 1/w_1 & 1/w_1 & 0 & 1/w_1 \\ 1/w_1 & 1/w_1 & 0 & 1/w_1 & 1/w_1 \\ 1/w_4 & 0 & 1/w_4 & 1/w_4 & 1/w_4 \\ 0 & 1/w_4 & 1/w_4 & 1/w_4 & 1/w_4 \end{vmatrix}$$

Proceeding as in previous section, we use the Lemma given in section III and apply lagrange multipliers [11] to A and get following cases.

1. *Case* 1 : If $3w_1 \geq 2w_4$ and $2w_4 \geq w_1$ then X^*=($w_1/(3w_1+2w_4)$,$w_1/(3w_1+2w_4)$,$w_1/(3w_1+2w_4)$, $w_4/(3w_1+2w_4)$,$w_4/(3w_1+2w_4)$) and
Y^*=(($3w_1$-$2w_4$)/($3w_1+2w_4$),($3w_1$-$2w_4$)/($3w_1+2w_4$), ($2w_4$-w_1)/($3w_1+2w_4$),($2w_4$-w_1)/($3w_1+2w_4$), ($2w_4$-w_1)/($3w_1+2w_4$)) are optimal strategies for PI and PII and val(A) is

$$\mathrm{val}(A) = \frac{4}{(3w_1 + 2w_4)}.$$

2. *Case* 2 : If $3w_1 < 2w_4$ then X^*=(1/3,1/3,1/3,0,0) and Y^*=(0,0,1/3,1/3,1/3) are optimal strategies for PI and PII and val(A) is

$$\mathrm{val}(A) = \frac{2}{3w_1}.$$

3. *Case* 3 : If $2w_4 < w_1$ then X^*=(0,0,0,1/2,1/2) and Y^*=(1/2,1/2,0,0,0) are optimal strategies for PI and PII and val(A) is

$$\mathrm{val}(A) = \frac{1}{2w_4}.$$

From these three cases, we get

$$\mathrm{val}(A) = \min\left((\frac{4}{3w_1 + 2w_4}), (\frac{2}{3w_1}), (\frac{1}{2w_4})\right).$$

Hence,

$$R^* = \min\left((\frac{3w_1 + 2w_4}{4}), (\frac{3w_1}{2}), (2w_4)\right). \tag{5.7}$$

Using (5.6) and (5.7), we have

$$\frac{R^{**}}{R^*} = \frac{\min\left(3w_1, w_1 + w_2, w_1 + w_4\right)}{\min\left((\frac{3w_1+2w_4}{4}), (\frac{3w_1}{2}), (2w_4)\right)} \tag{5.8}$$

If we have w_1=w_4=w then,

$$\frac{R^{**}}{R^{*}} = \frac{(2w)}{(\frac{5w}{4})} = \frac{8}{5} = 1.60. \tag{5.9}$$

5.3 Analysis for triple butterfly network

Figure 3 shows triple butterfly network. Here $w_i > 0$ denotes the link capacities of the edges in the graph.

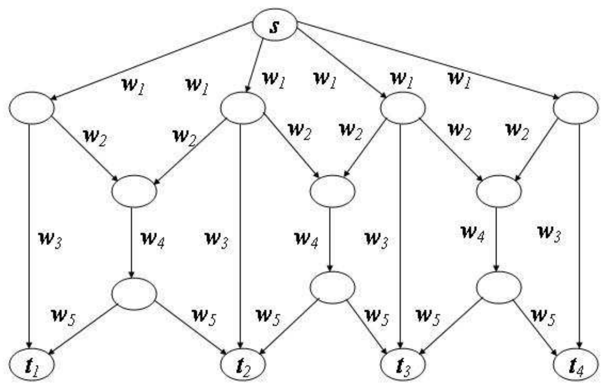

Figure 5.4: Triple-symmetric Butterfly network

For the above network taking into account assumptions as in case of singular butterfly network, R^{**} is equal the following seven values.

1. $4w_1$

2. w_1+w_2

3. w_2+w_3+$\min\{w_1, w_2\}$

4. w_1+w_4

5. $2w_4+w_3$

6. $2w_1+w_3+w_5$

7. $w_1+w_2+w_3+w_4$

From this set under the assumptions in section V, we only take minimum min-cuts and thus

$$R^{**} = \min\left(4w_1, w_1 + w_2, w_1 + w_4\right). \tag{5.10}$$

Using assumptions about link capacities stated earlier to compute R^{**}, we have all four links with capacities w_1 and all three links with capacity w_4 as dominating and these 7 links form 7 multi cast routes. After elimination of dominated links and multicast routes, we get the following payoff matrix.

$$A = \begin{vmatrix} 1/w_1 & 1/w_1 & 1/w_1 & 1/w_1 & 1/w_1 & 1/w_1 & 0 \\ 1/w_1 & 1/w_1 & 1/w_1 & 1/w_1 & 1/w_1 & 0 & 1/w_1 \\ 1/w_1 & 1/w_1 & 1/w_1 & 1/w_1 & 0 & 1/w_1 & 1/w_1 \\ 1/w_1 & 1/w_1 & 1/w_1 & 0 & 1/w_1 & 1/w_1 & 1/w_1 \\ 1/w_4 & 1/w_4 & 0 & 1/w_4 & 1/w_4 & 1/w_4 & 1/w_4 \\ 1/w_4 & 0 & 1/w_4 & 1/w_4 & 1/w_4 & 1/w_4 & 1/w_4 \\ 0 & 1/w_4 & 1/w_4 & 1/w_4 & 1/w_4 & 1/w_4 & 1/w_4 \end{vmatrix}$$

Proceeding as in previous section, we use the Lemma given in section III and apply lagrange multipliers [11] to A and get following cases.

1. *Case 1:* If $4w_1 \geq 3w_4$ and $3w_4 \geq 2w_1$ then X^*=($w_1/(4w_1+3w_4)$,$w_1/(4w_1+3w_4)$,$w_1/(4w_1+3w_4)$, $w_1/(4w_1+3w_4)$,$w_4/(4w_1+3w_4)$,$w_4/(4w_1+3w_4)$,$w_4/(4w_1+3w_4)$) and Y^*=(($4w_1$-$3w_4$)/($4w_1+3w_4$),($4w_1$-$3w_4$)/($4w_1+3w_4$), ($4w_1$-$3w_4$)/($4w_1+3w_4$),($3w_4$-$2w_1$)/($4w_1+3w$ ($3w_4$-$2w_1$)/($4w_1+3w_4$),($3w_4$-$2w_1$)/($4w_1+3w_4$),($3w_4$-$2w_1$)/($4w_1+3w_4$)) and val(A) is

$$\text{val}(A) = \frac{6}{4w_1 + 3w_4}.$$

2. *Case 2 :* If $4w_1 < 3w_4$ then X^*=(1/3,1/3,1/3,0,0,0,0) and Y^*= (0,0,0,$(w_4$-$w_1)/w_4$,$w_1/3w_4$,$w_1/3w_4$
   and val(A) is

$$\text{val}(A) = \frac{(3w_4 - w_1)}{3w_1 w_4}.$$

3. *Case 3 :* If $3w_4 < 2w_1$, then X^*=(0,0,0,0,1/3,1/3,1/3) and Y^* = (1/3,1/3,1/3,0,0,0,0)
   and val(A) is

$$\text{val}(A) = \frac{2}{3w_4}.$$

From these three cases, we get

$$\text{val}(A) = \min\left((\frac{6}{4w_1 + 3w_4}), (\frac{3w_4 - w_1}{3w_1 w_4}), (\frac{2}{3w_4}) \right).$$

Hence

$$R^* = \min\left((\frac{4w_1 + 3w_4}{6}), (\frac{3w_1 w_4}{3w_4 - w_1}), (\frac{3w_4}{2}) \right). \tag{5.11}$$

Using (5.10) and (5.11), we have

$$\frac{R^{**}}{R^*} = \frac{\min\left((4w_1, w_1 + w_2, w_1 + w_4)\right)}{\min\left((\frac{4w_1+3w_4}{6}), (\frac{3w_1 w_4}{3w_4 - w_1}), (\frac{3w_4}{2}) \right)}. \tag{5.12}$$

If we have w_1=w_4=w then,

$$\frac{R^{**}}{R^*} = \frac{(2w)}{(\frac{7w}{6})} = \frac{12}{7} = 1.72. \tag{5.13}$$

5.4 Analysis for generic butterfly network

Generic butterfly network is the singular-symmetric butterfly network repeated n times. So it has one source and a total of $n+1$ sinks.

Information rate with Network Coding (NC),i.e., R^{**} will be same as that of Triple network except one term of $(n+1)w_1$.

$$R^{**} = \min\left((n+1)w_1, w_1 + w_2, w_1 + w_4\right). \tag{5.14}$$

Using assumptions about link capacities stated earlier to compute R^{**}, we have $(n{+}1)$ dominant links with capacities w_1 and n dominant links with capacity w_4. Thus the payoff matrix for this network will be $(2n+1) \times (2n+1)$.

Graph theoretic approach cannot solve this pay off matrix. But one can give an intuitive observation based on the results from above three sections.

1. *Case* 1 : If $(n+1)w_1 \geq nw_4$ and $nw_4 \geq (n-1)w_1$ then $x_1{=}w_1/((n{+}1)w_1{+}nw_4)$,
   $x_2{=}w_1/((n{+}1)w_1{+}nw_4)$,...,$x_n{=}w_1/((n{+}1)w_1{+}nw_4)$; $x_{n+1}{=}w_4/(n{+}1)w_1{+}nw_4)$,..., $x_{2n+1}{=}w_4/($
   And $y_1{=}((n{+}1)w_1\text{-}nw_4)/((n{+}1)w_1{+}nw_4)$,
   $y_2{=}((n{+}1)w_1\text{-}nw_4)/((n{+}1)w_1{+}nw_4)$,...,
   $y_n{=}((n{+}1)w_1\text{-}nw_4)/((n{+}1)w_1{+}nw_4)$;
   $y_{n+1}{=}(nw_4\text{-}(n\text{-}1)w_1)/((n{+}1)w_1{+}nw_4)$,...,
   $y_{2n+1}{=}(nw_4\text{-}(n\text{-}1)w_1)/((n{+}1)w_1{+}nw_4)$.

From these conditions, it follows that

$$\text{val}(A) = \frac{2n}{((n+1)w_1 + nw_4)}$$

Hence,

$$R^* = \frac{((n+1)w_1 + nw_4)}{2n} \tag{5.15}$$

Using (5.14) and (5.15), we have

$$\frac{R^{**}}{R^*} = \frac{\min\left((n+1)w_1, w_1 + w_2, w_1 + w_4\right)}{\frac{((n+1)w_1 + nw_4)}{2n}} \tag{5.16}$$

If we have w_1=w_4=w then,

$$\frac{R^{**}}{R^*} = \frac{(4nw)}{((2n+1)w)} = \frac{4n}{2n+1}. \tag{5.17}$$

Chapter 6

Conclusion and Future Work

In this work, I have studied information network flow problem on a single source multi-cast network. I have tried to extend the idea in [6] by analyzing a generic butterfly network under suitable assumptions about link capacities. From the results, I could conclude that information rate due to Network coding is coming out to be same in all versions of the network. This means as number of nodes are increasing the network, no additional gain is coming out by using NC. On the other hand, information rate due to network switching is decreasing as we are duplicating singular butterfly network. Hence, simple switching for generic butterfly network does not increase throughput. Further, switching gain is equal to coding gain only when all link capacities are equal.

Application of game theory for network switching is not generic and requires certain conditions on link capacities. For the next semester, I will try device an optimal network switching strategy from min-cut trees for any generic communication point to point network doing information multi-cast. I would also formulate a strategy to find switching gap for a generic single source communication network. It would also be an interesting exercize to to study the effects for channel losses on such a network when operating in wireless scenario.

Bibliography

[1] Ahlswede, N. Cai, S.-Y. Li, and R. W. Yeung, "Network information flow," *IEEE Trans. Inform. Theory*, vol. 46, no. 4, pp. 12041216, July 2000.

[2] S.-Y. Li, R. W. Yeung, and N. Cai, "Linear network coding," *IEEE Trans. Inform. Theory*, vol. IT-49, no. 2, pp. 371381, Feb. 2003.

[3] Christina Fragouli, JeanYves Le Boudec, Jorg Widmer, "Network Coding: An Instant Primer," *LCA-REPORT-2005-010.*

[4] R. W. Yeung, S.-Y. Li, N. Cai, and Z.Zhang,"Theory of network coding," submitted to *Foundations and Trends in Commun. and Inform. Theory*, preprint, 2005.

[5] C.K Ngai and R. W. Yeung, "Network switching gap of combination networks," in 2004 *IEEE Inform. Theory Workshop*, 24-29, Oct 2004, pp. 283-287.

[6] Xue-Bin Liang, "On the Switching Gap of Ahlswede-Cai-Li-Yeung's Single-Source Multicast Network," 2006 *IEEE Int. Symp. Inform. Theory*, Seattle, Washington, USA, July 2006.

[7] Xue-Bin Liang, "Matrix Games in the Multicast Networks: Maximum Information Flows With Network Switching," *IEEE Trans. Inform. Theory*, vol. 52, no. 6, June 2006.

[8] G. Owen, *Game Theory*, 3^{rd} edition, San Diego: Academic Press, 1995.

[9] J. C. C McKinsey, *Introduction to the Theory of Games.* New York: McGraw-Hill Book Company, Inc., 1952, and Mineola, NY: Dover Publications, Inc., 2003.

[10] G.W Brown, "Iterative sollution of the games by fictious play." in *Activity analysis of production and Allocation*, T.C Koopmans, Ed. NewYork :Wiley, 1951, pp. 374-376.

[11] Dan Klein, "Lagrange Multipliers without Permanent Scarring," *Computer Science Division, University of California at Berkeley.* Available at www.cs.berkeley.edu/ klein/papers/lagrange-multipliers.pdf.

[12] R. J. McEliece, "Information multicasts," revised version of talk presented on Nov. 22, 2004 at *Lee Center meeting, California Institute of Technology, Pasadena, CA*. [online]. Available at http://www.ee.caltech.edu/EE/Faculty/rjm/papers/Lunch.pdf.

[13] J. Szep and F. Forgo, *Introduction to the Theory of Games.* Dordrecht: D. Reidel Publishing Company, 1985.