

Are there numbers, which equal their logarithms?

A computer-based activity

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Abstract

Problem solving and exploratory activities using technology as a tool continue to receive attention in recommendations for school mathematics. This article presents an experience in pre-college level type of teaching aiming at familiarizing students with the process of mathematical research. Students discovered a result based on a computer activity and verified the result analytically offering a rigorous proof.

1. Introduction

A primary goal for teaching and learning mathematics is the development of mathematical power for all students (1). In order to accomplish this goal, teachers must select interesting and intellectually stimulating mathematical tasks, and present opportunities for students to deepen their understanding of mathematics and its applications. Furthermore, they must promote the investigation of mathematical ideas, use technology to pursue these investigations, find connections to previous and developing knowledge, as well as employ cooperative learning experiences (2).

Open explorations are exciting and enriching experiences and can lead to some interesting findings. These activities develop confidence and self-esteem, challenge creative powers and give valuable insights into the nature and growth of mathematics. An investigation can begin in a variety of ways. It can start with an observation and an equation, with a search for patterns and relations, with an exploration of the implications of known properties and relations, and on a more sophisticated level, with a natural extension of topics that students have already encountered (3).

In this paper, a contextual exploratory research is presented. Our purpose is to give an example of an open exploration using logarithmic relations. Through this article, we seek to encourage strengthening of the links between the visual and the analytical. The computer graphical representation of a functional relation is conducive to immediate understanding and provides the motivation and basis for its mathematical analysis (4).

The following work took place with a group of senior level high school students (grade 12) in an advanced placement calculus course. These students had exhibited an interest and ability

in mathematics and had taken the prerequisite precalculus courses. All the students were college bound, between the ages of 17 and 18 and had selected university programs in science and mathematics. The course places emphasis on an intuitive understanding of the concepts of differential and integral calculus, encouraging students' observations and their formulation of conjectures, propositions and proofs.

2. Objectives of the research unit

Through discussion with the students on the benefits derived from the research unit, the following list of objectives was compiled:

- Allow students to experience the slow and deliberate process of developing and formulating mathematics.
- Allow students to investigate properties using technology, as well as make, test and refine conjectures and find counter-examples.
- Enhance student awareness of the necessity for a clear, concise and accurate statement of propositions, conjectures and results.
- Help students better understand that computer-based discovery activities in a calculus course should be followed by rigorous proofs of the conjectures developed during these activities.
- Help students to gain experience in conjecturing and in constructing simple valid arguments that mirror their logical thinking.
- Promote independent thought and creativity.
- Encourage cooperation and sharing of ideas.

3. The research question and classroom procedures

After a unit about curve sketching by means of the derivatives, we began with a review of some properties of exponential and logarithmic functions. The graphs of these functions allow the student to visualize relationships and determine important properties concerning domain and range, maxima and minima, asymptotes, intervals of increase or decrease, and concavity.

The question used to launch the research process was: 'Are there numbers which equal their logarithms?' It is our opinion that the initial research question should ideally be one to which neither the teacher nor any of the students has an immediate answer. In the typical classroom, the learning potential of the student is decided before he/she steps into the classroom. The teacher knows exactly what will be covered and time is not allowed for original considerations. Many ideas that the students are asked to prove are known to be true and, therefore, the process of such proof is likely to remain meaningless and purposeless. Often the way we teach mathematics leads students to believe that they learn mathematics by watching us solving problems and then mimicking what we do. Therefore, students need to be asked to prove mathematical assertions that are not known by them to be true with the teacher playing the risk-taker role. In other words, students should be expected to prove non-traditional conjectures that can be explored in an appropriate classroom environment. In this research process, the teacher has no guarantee that any mathematical results will emanate. The students could be disillusioned by the entire experience but, more likely, what they will learn about the research process will be rewarding in and of itself.

The above question can reinforce the students' concept of the graphical representations of the logarithmic functions and lead them to explore and discover a variety of interesting mathematical relationships. My calculus students had previously studied the derivative and its applications, so this problem presented an opportunity to review a broad range of precalculus as well as calculus topics.

Although the initial question was posed by the teacher, the students made the major mathematical contributions. Working individually or in groups, at home or at school, the students attempted to tackle the question in any way that they could. During a first-class session, the students were working in pairs in a computer/classroom laboratory when they began the exploration by investigating graphs, making observations, collecting data, analyzing results and trying to reach conclusions. In a second-class session, the students tried to offer some proofs of their conjectures and hypotheses.

During the exploratory process the students made the major mathematical contributions. The main task of the teacher was to encourage the students to pursue the exploration, even if they felt they were in the dark, by creating a class atmosphere in which the students felt free to ask questions and seek help, and were allowed to speculate, hypothesize and make errors without embarrassment.

4. Description of the research process

Let us begin the analysis of this problem and see what unfolds. First, we helped our students to formulate the problem algebraically. We tried to find a number $x > 0$ such that

$$\log_k x = x, \quad \text{where } k > 0, k \neq 1 \text{ is a constant} \quad [1]$$

In the first place, we decided to investigate the question of logarithms to the base e , i.e. natural logarithms. In this case, $k = e$, we look for $x > 0$ satisfying the logarithmic equation $\ln x = x$.

The students used computer-graphing utilities to graph the functions $y = \ln x$ and $y = x$. They soon saw that the two graphs do not intersect (Fig. 1). Thus, they easily concluded that there are no numbers that equal their logarithms to the base e .

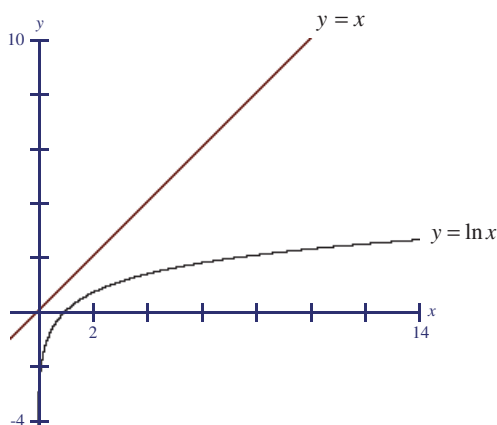


Fig 1. Graphs of $y = x$ and $y = \ln x$.

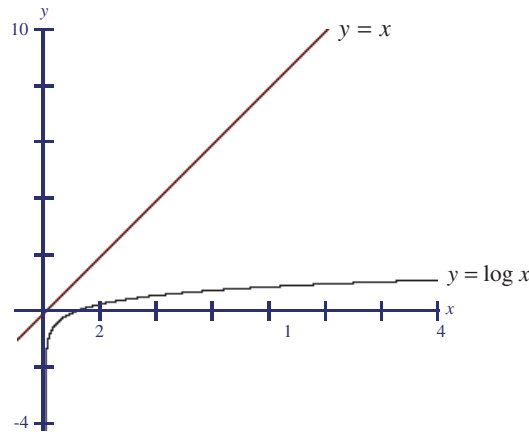


Fig 2. Graphs of $y = x$ and $y = \log x$.

Then I asked the students to repeat the same process with common logarithms (base 10). Working in the same way, they discovered that the two graphs $y = \log x$ and $y = x$ do not intersect either (Fig. 2). So there are no numbers that equal their logarithms to the base 10.

Afterwards, we decided to investigate the research question in the general case. The students plotted the line $y = x$ and began to encounter graphs of logarithmic functions to several bases. The students were encouraged to try several graphs, to collect and record data and make observations. This activity excited their curiosity and they quickly generated accurate data using the software to draw multiple graphs and zoom in to approximate values. The students drew several logarithmic curves for different base values and they calculated intersection points to a nicety.

After trying several different base values for k , students quickly realized that there are intervals such that, for any base value in this interval, the two equations intersect once, twice, or not at all. After some discussion we proposed that the students first search for logarithmic functions with a base value in the interval $(0, 1)$. Each pair of students selected several different base values and they saw that for any base value k , $0 < k < 1$ the graphs of the two equations $y = \log_k x$ and $y = x$ intersect once (Fig. 3). Therefore, for every base k , where $0 < k < 1$, there is always a number which equals its logarithm to this base.

Next, we encouraged our students to explore base values in the interval $(1, +\infty)$. Figure 4 shows the graphs when $k = 1.2, 1.3, 1.4, 1.5, 1.6, 1.7$. The students observed that, as the value of k decreases, the logarithmic functions get closer to the line $y = x$.

Figure 4 demonstrates that the first time the functions intersect on this interval is somewhere between $k = 1.4$ and 1.5 .

At this point we decided to select base values in the interval $(1.4, 1.5)$ in order to find the base value at which exactly one solution exists. Using the 'zoom in' button in the graphics program, we found the approximate base value of 1.4445 as the value where the two functions intersect once (Fig. 5).

Using the black cross cursor to zoom in on the point of intersection, we found the coordinates of this point to be approximately $\begin{cases} x = 2.7183 \pm 0.0082 \\ y = 2.7294 \pm 0.0083 \end{cases}$ (Fig. 6). These values

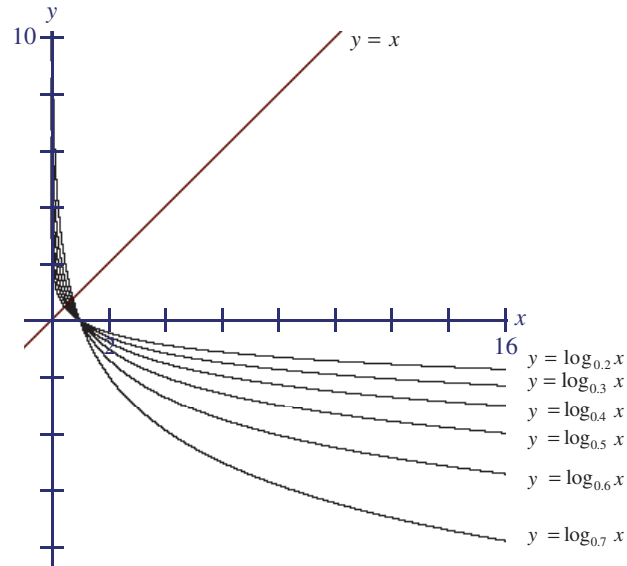


Fig 3. Graphs of $y = x$ and $y = \log_k x$ for $0 < k < 1$.

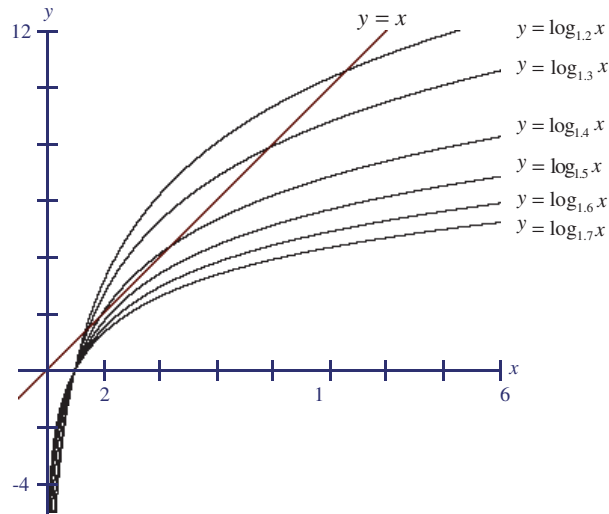


Fig 4. Graphs of $y = x$ and $y = \log_k x$ for $k > 1$.

are very close to the approximate value of the number e (≈ 2.71828). The students conjectured that the intersection point of the line $y = x$ and the logarithmic function with the base k is the point (e, e) . Indeed, substitution into the equation [1] gives:

$$\log_k e = e \iff k^e = e, \quad \text{thus } k = e^{1/e} \approx 1.4445.$$

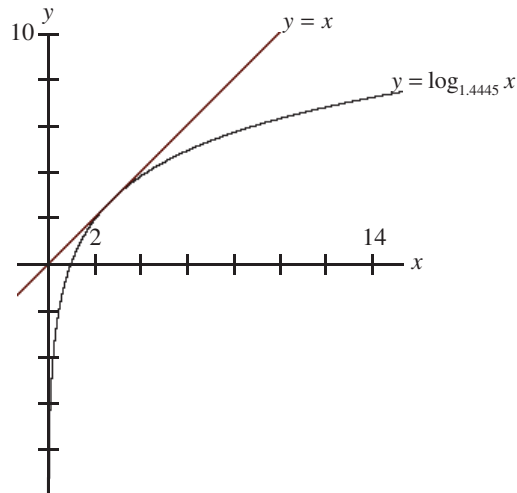


Fig 5. Graphs of $y = x$ and $y = \log_k x$ for $k = 1.4445$.

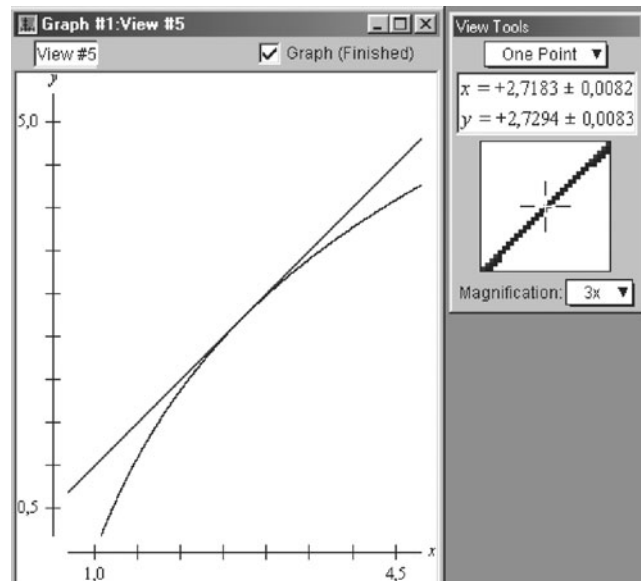


Fig 6. Zoom in on the point of intersection.

Therefore, the students quickly came to the conclusion that:

- For every k such that $0 < k < 1$ there is one intersection.
- For every k such that $1 < k < e^{1/e}$ there are two intersections.
- For every k such that $e^{1/e} < k < +\infty$ the two graphs do not intersect.
- If $k = e^{1/e}$ the two graphs intersect once at $x = k = e$.

In the discussion that followed, students summarized the results, answering the initial research question:

- For every k in the interval $(0, 1)$ there is a number, which equals its logarithm to the base k .
- For every k in the interval $(0, e^{1/e})$ there are two numbers, which equal their logarithms to the base k .
- If $k = e^{1/e}$ the number e equals its logarithm to the base $e^{1/e}$.
- If $k > e^{1/e}$ there are no numbers, which equal their logarithms.

In this activity students discovered a result on the basis of the pictures drawn on their computer screens. The students realized that they were drawing general conclusions from several examples. They also knew that inductive reasoning is not infallible, so they had to attempt and formulate a deductive proof. Deciding whether conjectures are true motivates students to discuss proof and to decide on the evidence required to convince themselves and the class of their results (5). It is a great compliment to the teacher to have students develop their own mathematics and be eager to see if a proof exists for their conjectures. Deductive reasoning is a fundamental instrument of mathematical thinking, and obviously a key point in the teaching and learning of mathematics (6).

The students were motivated to prove this result formally because they had predicted its conclusion first. By general agreement, we decided that we should be able to accomplish this task. As a home assignment, the students individually tried to prove the above statement. Then, during the second part of the activity, some of them presented their results to their classmates and all of them were challenged to discuss them. What follows is a proof for the general case, which should be regarded as a contribution of the whole class.

5. The proof

Let

$$f(x) = x - \log_k x = x - \frac{\ln x}{\ln k} = \frac{x \ln k - \ln x}{\ln k}, \quad x > 0, k > 0, k \neq 1.$$

Then,

$$f'(x) = 1 - \frac{1}{x \ln k} = \frac{x \ln k - 1}{x \ln k}.$$

If $0 < k < 1$, then $\ln k < 0$ and $f'(x) > 0$. So f is increasing on the interval $(0, 1)$. Since $\lim_{x \rightarrow 0+} f(x) = +\infty$ and $f(1) = 1$, then the range of f is $f(0, 1) = (-\infty, 1)$. Hence, there is one number $x_0 \in (0, 1)$ such that $f(x_0) = 0$, i.e. $x_0 = \log_k x_0$. Therefore, for each k in $(0, 1)$ there is one number which equals its logarithm to the base k .

If $1 < k < e^{1/e}$, then $f'(x) = 0 \iff x = 1/\ln k$. Hence, the point $x = 1/\ln k$ is a critical point. The corresponding value of $f(x)$ is

$$f\left(\frac{1}{\ln k}\right) = \frac{1}{\ln k} - \frac{\ln(1/\ln k)}{\ln k} = \frac{1 + \ln(\ln k)}{\ln k}. \quad [2]$$

Since

$$1 < k < e^{1/e}, \quad 0 < \ln k < \frac{1}{e}, \quad \ln(\ln k) < -1 \quad \text{and} \quad 1 + \ln(\ln k) < 0. \quad [3]$$

From [2] and [3], it follows that $f(1/\ln k)$ is a ratio of a negative number and a positive number, whence

$$f\left(\frac{1}{\ln k}\right) < 0. \quad [4]$$

The table of change for the function f is as follows:

	x	0	$\frac{1}{\ln k}$	$+\infty$
	f'		$-$	$+$
	f			

$f_{\min} = f\left(\frac{1}{\ln k}\right) < 0$

Since $\lim_{x \rightarrow 0+} f(x) = +\infty$ and f is decreasing on $A = (0, (1/\ln k)]$, then the range of f is $f(A) = [f(1/\ln k), +\infty)$. Since $0 \in [f(1/\ln k), +\infty)$, there is one number $x_1 \in (0, (1/\ln k))$ such that $f(x_1) = 0$ or $x_1 = \log_k x_1$.

In the same way

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \left(x - \frac{\ln x}{\ln k} \right)^{(\infty - \infty)} = \lim_{x \rightarrow +\infty} x \left(1 - \frac{\ln x}{x \ln k} \right) = \lim_{x \rightarrow +\infty} x \lim_{x \rightarrow +\infty} \left(1 - \frac{\ln x}{x \ln k} \right) \quad [5]$$

$$\lim_{x \rightarrow +\infty} \left(1 - \frac{\ln x}{x \ln k} \right)^{(\infty/\infty)} = \lim_{x \rightarrow +\infty} \left(1 - \frac{1}{x \ln k} \right) = 1. \quad [6]$$

From [5] and [6] we obtain $\lim_{x \rightarrow +\infty} f(x) = +\infty$. Since f increases on the interval $[(1/\ln k), +\infty)$ the range of f is the interval $[f(1/\ln k), +\infty)$. Since $0 \in [f(1/\ln k), +\infty)$, there is one number $x_2 \in ((1/\ln k), +\infty)$ such that $f(x_2) = 0$ or $x_2 = \log_k x_2$.

Hence, we verified that for each k in the interval $(e, e^{1/e})$ there are two numbers $x_1 \in (0, (1/\ln k))$ and $x_2 \in ((1/\ln k), +\infty)$ which equal their logarithms to the base k .

If $k = e^{1/e}$, then $\ln k = 1/e$, $f(e) = [e \cdot (1/e) - 1]/e \cdot (1/e) = 0$ and thus $e = \log_{e^{1/e}} e$. Hence, there is the number e , which equals its logarithm to the base $e^{1/e}$.

If $e^{1/e} < k < +\infty$, then $f(1/\ln k) > 0$ and $f(x) > f(1/\ln k) > 0$, which yields $x > \log_k x$. Hence, there are no numbers that equal their logarithms, and the proof is completed.

6. Concluding remarks

Looking at the substantial changes that technology has brought in recent years, we believe that instruction in mathematics will have to catch up with the new circumstances or else will become increasingly irrelevant.

This exploration is one example of how using the technology in a mathematics class allows students to discover and explore mathematical relationships. In this nice activity, students discovered a result based on a computer activity and then verified the result analytically. The main questions were posed and solved interactively by the student and the computer.

It must be stressed that this research effort did not add new facts to the world of mathematics. That was not our purpose. The point was that the knowledge gained was sometimes new to the teacher as well as the learner. These findings do not exist in common calculus textbooks that students use and therefore they had the chance to rediscover them enjoying the unique experiences hidden in an adventure.

The consideration of equation [1] in this class led to a great deal of mathematical thinking and to the joy of discovery. It allowed us to emphasize that technology suggests quick, informal-visual solutions, which should be regarded as stepping-stones to additional investigation and rigorous proof.

The principal purpose of teaching high school mathematics is not the transformation of students' intuitive way of thinking and learning to an analytico-logical one, but their combination and harmonious development (7). Explorations of calculus concepts designed as guided activities in the way described above seem to be a very suitable means of achieving the main goals of mathematics education. Technology has been proved to enhance and enrich mathematics for all students. It also promises to help students learn what many consider to be the absolute heart and soul of mathematics—reasoning and proof.

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