Expanding in-service mathematics teachers' horizons in creative work using technology

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This article describes the experiences from a seminar in the teaching of mathematical reasoning and problem solving designed to prepare in-service high school mathematics teachers to teach genuine mathematical activity in a computer-based environment. Presented with a set of unfamiliar tasks and activities, the participants were encouraged to investigate each of them, using the Geometer's Sketchpad software, and then to justify their assertions accordingly. In the exploratory process the student teachers make the major mathematical contributions while the teacher plays the role of facilitator. The mathematics teachers began to realize the power of technology in teaching mathematics and were pleased to return to their classrooms with a great number of experiences and ideas for immediate use.

1. Introduction

In mathematics education the contribution of great mathematicians and educators as Polya [1,2] or philosophers of mathematics as Lakatos [3] promoted discovery approaches in mathematics teaching. As a consequence, it is increasingly emphasized that, in all grades, pupils should be given more opportunity to experience typical processes of mathematical activity, like looking for patterns, making generalizations or specializations or analogies, conjecturing or guessing, proving, etc. In this way, students might develop a stronger feeling that mathematics is a living, ever-growing, open subject and that mathematical activity is a dynamic, powerful and worthwhile endeavor as opposed to the long tradition according to which mathematics has been presented as a ready-made prefabricated body of knowledge.

The reasoning skills that all secondary students should know include making and testing conjectures, formulating counterexamples, following logical arguments, judging the validity of arguments, and constructing simple valid arguments [4,5]. Making and testing conjectures are at the heart of proof in mathematics. Mathematical proofs and conjectures are central to mathematics and mathematics education. The most significant potential contribution of proof in mathematics education is the contribution proofs might make to students' mathematical understanding [6].

Therefore, it is reasonable to argue that students should be exposed to, and preferably experience for themselves, the search for patterns, the tentative generalizing, the further search for confirming or counter-examples and the hypothesizing of a result, which in actual fact comprises a large part of the professional mathematicians' work. Ideally, they should then continue and confirm the result by proving it, or seeing it proved deductively as the mathematical culture and mathematical tradition calls for.

Technology can play a powerful role in this process of making, evaluating, and refining conjectures as well as in other aspects of learning mathematical reasoning. Computer construction programs, such as the Geometers' Sketchpad [7], and Cabri Geometry [8], allow students to experiment; to investigate algebraic and geometric properties; to make, test and refine conjectures; and to find counter-examples.

However, the density of any such efforts to improve as well as to cultivate students' mathematical reasoning by means of technology lies in the hands of the classroom teacher; in fact, the success of any curricular reforms in mathematics ultimately depends on classroom teachers [9]. Hence, genuine mathematical activity as, for example, problem solving or problem posing will become the central focus only if the teacher recognizes its importance and fosters a classroom environment, which is conductive to exploration, inquiry, reasoning, and communication. "Teachers play an important role in the development of students' problem-solving dispositions by creating and maintaining classroom environments, from pre-kindergarten on, in which students are encouraged to explore, take risks, share failures, and question one another" ([5], p.36).

George Polya [10] expressed the point of view that the teaching of teachers of mathematics should offer experience in independent, creative work, at an appropriate level, through a problem- solving seminar or through any other suitable medium.

2. Purpose and methodology

The purpose of this article is to describe some snapshots of a two-month seminar in the teaching of mathematical reasoning and problem-solving designed to prepare in-service high school teachers of mathematics to teach mathematical activity as a process rather than a product of learning in a technology environment.

As specific objectives I put some of the advantages that discovery approaches offer to the teaching of mathematics in a technology environment:

1. Facilitate students' development of an attitude towards learning and inquiry, towards guessing and hunches, towards the possibility of solving problems on their own.
2. Allow students to experience the exciting process of developing mathematics.
3. Promote independent thought and creativity and encourage the sharing of ideas.
4. Alter students' perspective from mathematics as a subject in which everything is right or wrong, to mathematics as a discipline of modifying and changing until a convincing justification has been found.
5. Encourage students to work in small groups, to collect and organize data, to conjecture about the results, to listen to the thoughts of peers, to contribute to a collective effort, and to practice communication skills.

The seminar was organized and conducted by the author, who as a school advisor is charged with the responsibility of delivering continuing education to mathematics teachers to improve classroom instruction, and took place in the Center for In-service Education of Patras, Greece. The participants were twelve in-service high school mathematics teachers (two female and ten male) in grades 9-12. All of them had a thorough background in mathematics, but little direct experience with the use of technology to teach mathematics. For seminar participants, and for many other mathematics teachers, their own mathematics backgrounds are hardly associated with discovery learning. Rather, what comes to mind is didactic activity, e.g., required theorems often memorized without sufficient understanding, model problems reconstructed by rote, textbook exercises performed repetitiously.

The seminar offered the participants a totally new experience in creative work with opportunities to formulate, communicate and support original mathematical conjectures. The participants used a computer-based version of Sketchpad(4). It is also possible to do the same explorations on graphics calculators, such as the TI-92. Student teachers were given rudimentary explorations earlier in the semester to familiarize them with the mechanics of the software.

Presented with a set of ten tasks and activities dealing with the concepts of algebra, geometry and calculus, the participants were asked to investigate each of them and then to justify their assertions accordingly. The tasks were chosen to be unfamiliar in their specifics, not commonly seen in textbooks, yet based on traditional mathematical concepts. What is most essential is that for many of the participants this was the first time they were being asked to investigate a mathematical statement without knowing in advance whether the statement is true or not. This was a truly new dimension in mathematical inquiry for these high school mathematics teachers.

The student teachers worked in pairs at computers during a 3-hour session weekly. Ideally they should come up with their conjectures and share with the whole group. Once the students' conjectures were understood by the whole group, they could either find counter-examples or prove the conjectures.

When a student teacher had a conjecture about the question under consideration, it was presented to the rest of his/her colleagues. Next, the class tried to search for exceptions to the conjecture. If one such was found, the proposition was modified accordingly. If none was found, an attempt was made at its proof, which was mostly not completed in the class but was developed as a result of the contributions of a number of student teachers over a period of several days.

In such a process the instructor must play the principal role of a facilitator. He/she encourages the students to pursue the exploration even if they feel they are in the dark. The main task of the instructor as facilitator is to encourage students' formulation of propositions, conjectures and proofs, creating a class atmosphere in which the students feel free to ask questions and seek help, and are allowed to speculate, hypothesize, and make errors without embarrassment. The instructor also promotes a climate in which he/she appears as a guide and counselor whose main task is to stimulate students' activity and learning.

3. The presentation of samples activities

The experiences of the author and the findings from three such activities are presented in the following.

Activity 1.

In 1899, Frank Morley, came across a result so surprising that it entered mathematical folklore under the name of "Morley's Miracle". Start with an arbitrary triangle ABC. Trisect the angles in $\triangle ABC$ and let D be the point of intersection of the two trisectors closest to side BC with E and F defined similarly. Morley's marvelous theorem states that $\triangle DEF$ is equilateral. Investigate further the Morley's configuration. Especially search for a relation between the six segments AF, AE, BF, BD, CD and CE (figure1).

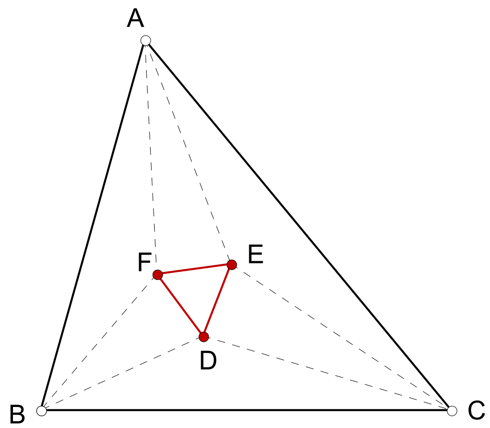

Figure 1: The Morley's configuration

It is well known that it is impossible to trisect an arbitrary angle using straightedge and compass. Instead the Geometer's Sketchpad is able to trisect an arbitrary angle by computing the one third of its measure.

Having constructed a dynamic triangle with trisections, which can be dragged, the numerical dada on the measures of the straight segments, which connect the vertices of the initial triangle ABC with the equilateral triangle DEF, is displayed by the program.

The dynamic software allowed students to gather data from several different examples quickly and accurately, a task that would be daunting if attempted by pencil-and-paper methods. Patterns evident among all examples can then be explored.

After several trials, a conjecture was formulated that $AE \cdot BF \cdot CD = AF \cdot BD \cdot CE$. In this situation, two students used the calculator embedded in the geometry program to multiply the straight segments AE, BF, CD and the straight segments AF, BD and CE. Using the Tabulate function of the Geometer's Sketchpad, the students recorded in a table the product of the three segments $AE \cdot BF \cdot CD$ and the product of the other three segments $AF \cdot BD \cdot CE$ as point A moves (figure 2).

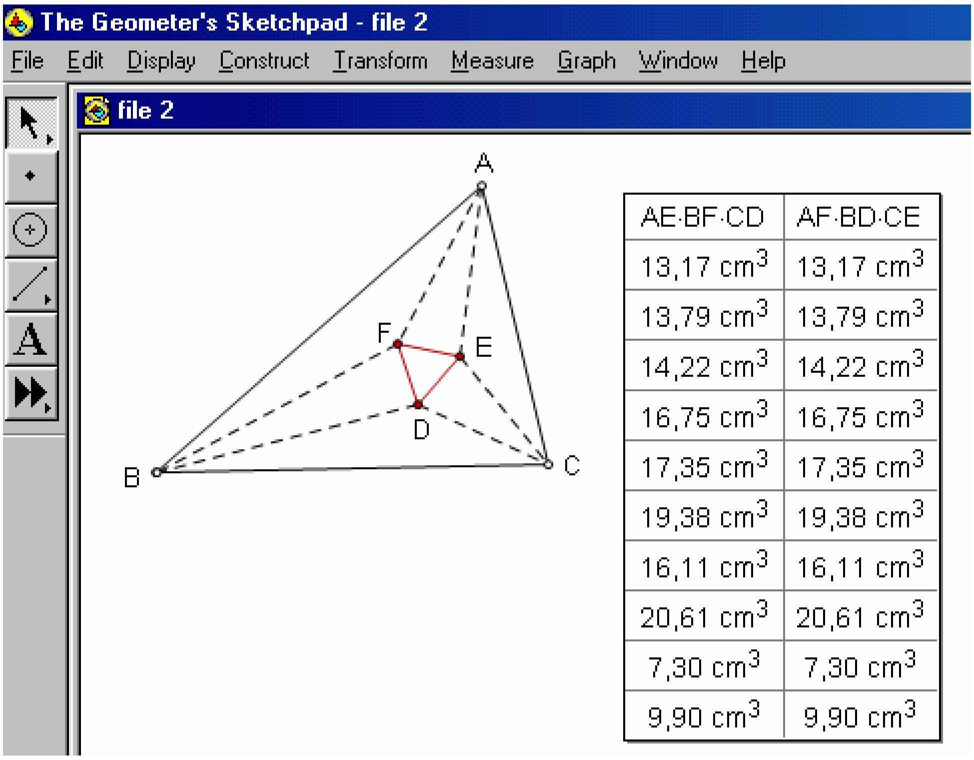

Figure 2

Soon they demonstrated that this relationship seemed to hold with every distortion of the initial triangle ABC, supplying strong evidence that the conjecture was true. This conjecture was thrown open to the class, and, before a proof was found, all the groups of student teachers found, by exploring the distortions of the initial triangle ABC on dynamic software, that the conjecture seemed to be true, without any doubt. When technology is used in this manner, the instructor has to be confident enough to let the students' inquiry guide them to investigate if a proof exists for their conjectures.

At this point some teachers expressed the concern that high school students may believe proofs are unnecessary. As one participant stated, "Why would my students want to bother proving this proposition when they see that it is true, right in front of them?". However, when asked explicitly if the above evidence constitutes a proof, all of them expressed the belief that a formal proof is different from proof by many examples. Although computer software provides compelling evidence of the truth of a theorem, it is certainly not a proof. Whether or not one studies mathematics with a computer, deductive proof is still crucial for several reasons. One important reason to prove theorems is that a proof can explain why a theorem is true [11].

The Geometer's Sketchpad is an example of software that gives a wealth of opportunities to discover relationships, to show when statements are false, and to give intuitive understanding, although it is not competent for developing general proofs. At this stage the students set aside the computer and tried to develop a rigorous proof. In the next session several of them offered the following proof:

Let the angles at A equal α, the angles at B equal β and the angles at C equal γ (figure 3). Then, applying the low of sines to the triangle ABF yields

$$\frac{AF}{\sin\beta} = \frac{BF}{\sin\alpha} \quad \text{or} \quad \frac{AF}{BF} = \frac{\sin\beta}{\sin\alpha}$$

(1)

In the same way, applying the low of sines to the triangles BDC and AEC we have

$\frac{BD}{CD} = \frac{\sin\gamma}{\sin\beta}$ (2) and $\frac{CE}{AE} = \frac{\sin\alpha}{\sin\gamma}$ (3) respectively.

Multiplying (1), (2) and (3) we get $\frac{AF}{BF} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = \frac{\sin\beta}{\sin\alpha} \cdot \frac{\sin\gamma}{\sin\beta} \cdot \frac{\sin\alpha}{\sin\gamma} = 1$.

Therefore, $AF \cdot BD \cdot CE = BF \cdot CD \cdot AE$

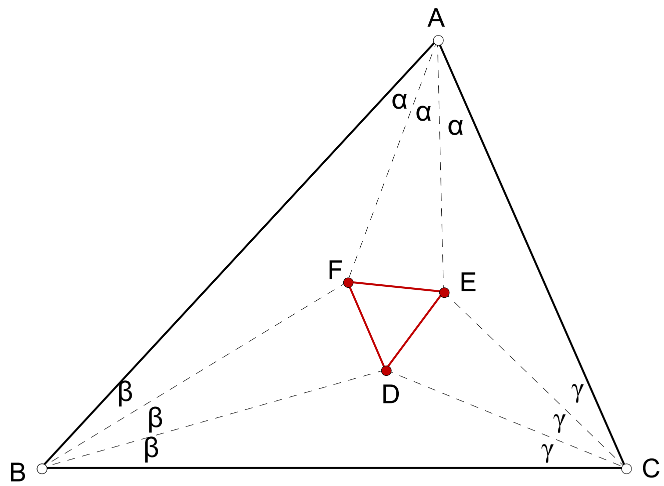

Figure 3

Activity 2

In the Morley's configuration investigate the relation between the area of the initial triangle ABC and the area of the equilateral triangle DEF.

Using the dynamic software the students, first, calculated the areas of the two triangles and tried to explore how the ratio $\frac{(ABC)}{(DEF)}$ of areas was changed with every distortion of the initial triangle ABC.

After several experiments they could not discover any such relation, so some of them proposed to explore the special case when the initial triangle ABC is equilateral. In this situation the software showed that $\frac{(ABC)}{(DEF)} = 28.615$ and this value seemed to hold with every distortion of the equilateral triangle ABC (figure 4).

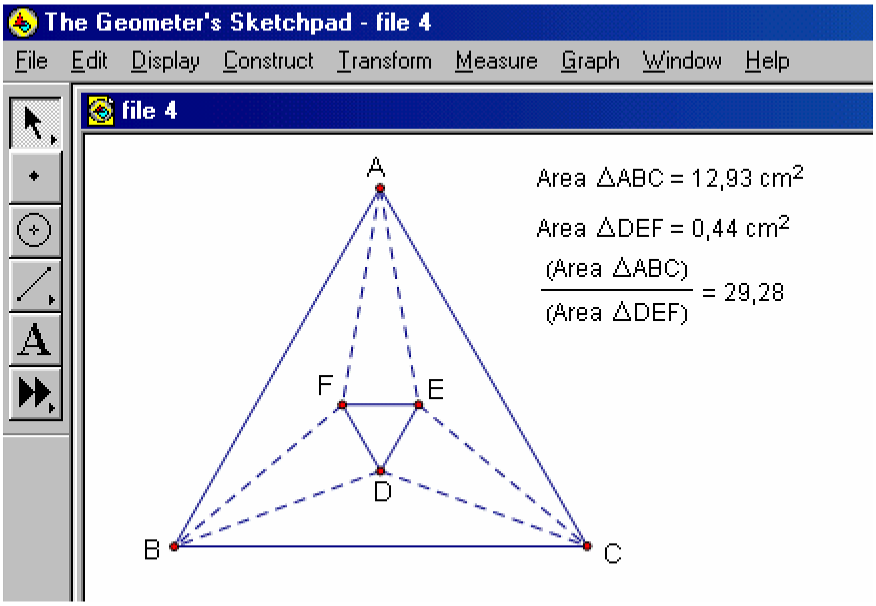

Figure 4

But why does the above ratio remain constant? How is this fact the consequence of other familiar results? These questions establish a very good reason to pursue a proof.

The class spent the last thirty minutes of a session trying to prove this conjecture but could not come up with a proof. To nurture open mathematical investigative minds in the students, the instructor discussed their failures to prove the statement and giving some suggestions the proof was assigned as homework for the next session.

In the following lines we present two different approaches for the proof, which are based on ideas of several students.

1st proof:

In this approach the law of sines is used as before. Since each of the trisected angles has a measure of 20^{o}, we see that $\angle AFB = 140^{o}$ and $\angle AEF = 80^{o}$ (figure 4).

Thus from the law of sines we obtain $\frac{AF}{\sin 20^{o}} = \frac{AB}{\sin 140^{o}}$ and $\frac{EF}{\sin 20^{o}} = \frac{AF}{\sin 80^{o}}$ respectively. It follows that $\frac{AB}{EF} = \frac{\sin 80^{o} \cdot \sin 140^{o}}{\sin^2 20^{o}}$.

Since the ratio of areas of similar triangles ABC and DEF equals the square of the ratio of corresponding sides, the desired ratio is

$$\frac{\sin^2 80^{o} \cdot \sin^2 140^{o}}{\sin^4 20^{o}} \approx 29.284.$$

2^{nd} proof:

This proof is based on the fact that the side DF in the Morley's equilateral triangle DEF is given by the expression DF = 8Rsinαsinβsinγ, where A = 3α, B = 3β, C = 3γ and R is the radius of the circle circumscribed around $\triangle ABC$ [12].

In our case we have $DF = 8R\sin^3 20^{o}$. Since in an equilateral triangle $R = \frac{a\sqrt{3}}{3}$, where a is its side, we have that

$$\frac{(ABC)}{(DEF)} = \frac{(AB)^2}{(DF)^2} = \frac{(AB)^2}{64R^2\sin^6 20^{o}} = \frac{3(AB)^2}{64(AB)^2\sin^6 20^{o}} = \frac{3}{64\sin^6 20^{o}} \approx 29.284$$

Activity 3.

> Start by sketching the graph of the function $f(x) = x^3 - 3x^2 - x + 3$. Find the zeros of f(x) and then sketch the graph of the tangent line through the average of two zeros. Make your observations.

Students easily found that f(x) = (x+1)(x-1)(x-3) and the three zeros are x = -1, x = 1, and x = 3. They then computed the average of the two zeros –1 and 1 to be 0 and they found the equation of the tangent line at the point (0,f(0)) to be y = -x + 3. Using the graph menu of Sketchpad they drew the tangent line and the result, shown in figure 5, was a surprise to everyone. The x-intercept of the tangent line that was created was x = 3, i.e. the third zero of f(x).

Students repeated the same process with two other zeros of the cubic polynomial f(x) and they found again that the tangent line drawn at the average of two zeros has an x-intercept equal to the third zero.

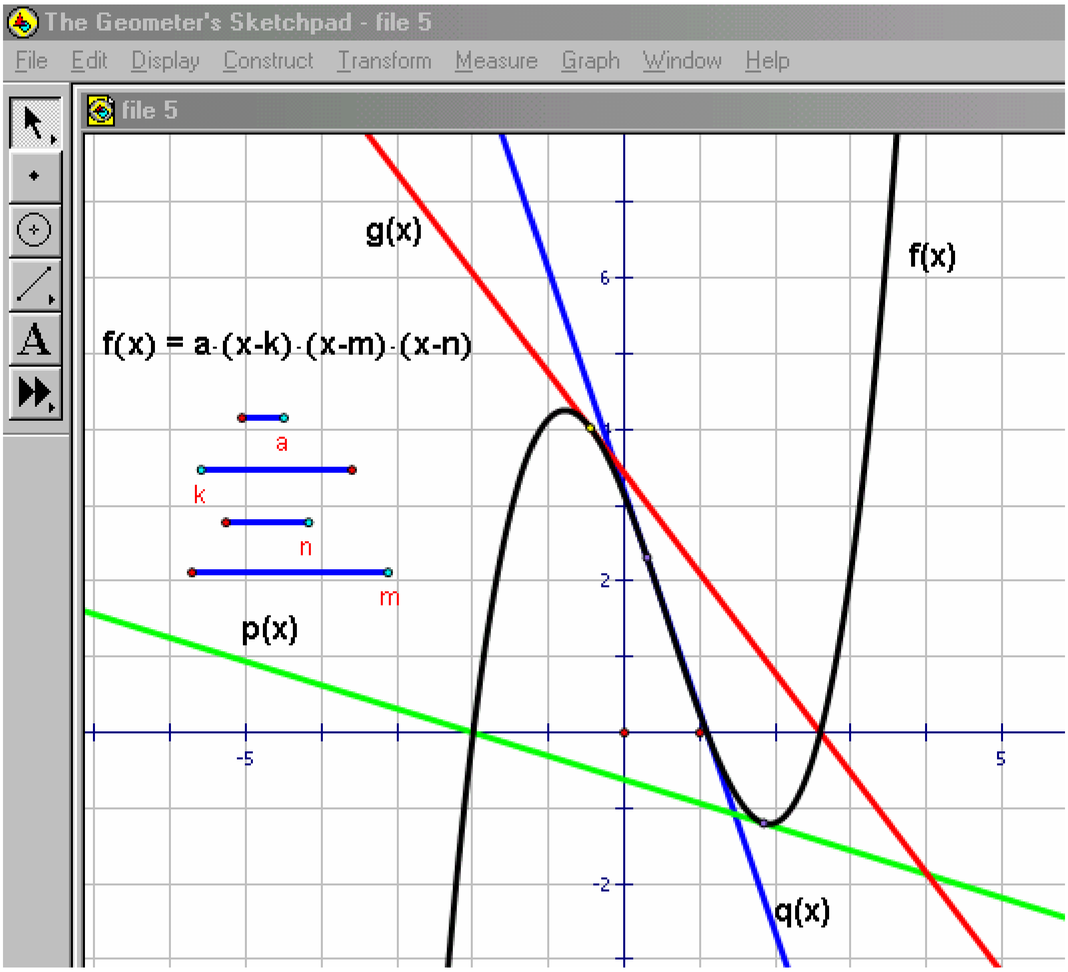

Figure 5

At first most of the students thought that this result might be simply a coincidence. Since the Geometer's Sketchpad is a dynamic program, we created a cubic polynomial of the general form f(x) = (x-k)(x-m)(x-n), setting up sliders for k, m and n. In this manner one can slide a point back and forth to control the values of three zeros k, m and n. At the same time, because the function is made into a directly manipulable object that can be translated, squeezed and stretched, the student has a rich opportunity to develop intuitions about these objects and the effects of different procedures on them.

Students created the tangent lines at the points $\left(\frac{k+m}{2}, f\left(\frac{k+m}{2}\right)\right), \left(\frac{k+n}{2}, f\left(\frac{k+n}{2}\right)\right)$ and $\left(\frac{m+n}{2}, f\left(\frac{m+n}{2}\right)\right)$, and they found out that these lines have an x-intercept equal to the third zero of the polynomial.

Next they repeated the same process changing all three zeros, k, m, and n. On the basis of the pictures drawn on their computer screens, students soon formulated the following conjecture:

The tangent line drawn to the cubic polynomial at the average of two zeros, passes through the third zero of the polynomial.

Students then were on a quest to determine why they obtained this result, and to explain why this result holds true for cubic polynomials. In the discussion that followed, some students expressed interesting ideas on how one could prove this result. Although they made a considerable effort, no one could give a full proof by the end of the session. In the next session five students had proved the above theorem. The following proof is a combination of three of these proofs.

Let us take a cubic with the roots 2k, 2m, and 2n, f(x) = (x-2k)(x-2m)(x-2n). Then $f'(x) = (x-2m)(x-2n) + (x-2k)(x-2n) + (x-2k)(x-2m)$ and after some computation $f'(x) = 3x^2 - 4(k+m+n)x + 4(kn+km+mn)$.

The average of two zeros 2k and 2m is k+m. The slope of the cubic at x = k+m is

$f'(k+m) = 3(k+m)^2 - 4(k+m+n)(k+m) + 4(kn+km+mn)$,

$f'(k+m) = 3k^2 + 3m^2 + 6km - 4k^2 - 8km - 4m^2 - 4kn - 4mn + 4kn + 4km + mn$,

$f'(k+m) = -k^2 - m^2 + 2km$, $f'(k+m) = -(k-m)^2$.

Next we find f(k+m) = (m-k)(k-m)(k+m-2n), f(k+m) = $-(k-m)^2(k+m-2n)$.

The equation of the tangent at the point $(k+m, f(k+m))$ is

$y = f'(k+m)(x-k-m) + f(k+m)$. Substituting, therefore, we have

$y = -(k - m)^2(x - k - m) - (k - m)^2(k + m - 2n)$.

Setting y = 0 to find the x-intercept yields $(k - m)^2(x - k - m) = -(k - m)^2(k + m - 2n)$ or $(k - m)^2x = -(k - m)^2(k + m - 2n) + (k + m)(k - m)^2$. Dividing both sides by $(k - m)^2$ we obtain x = - (k + m – 2n) + (k +m), x = 2n.

Therefore, the tangent line has its x-intercept at the third zero, which completes the proof.

4. The rest of activities

Next, we present the rest seven open-ended, computer-based activities, which were given to the participants in the seminar, to give the reader a clear sense of the type of task the participants were asked to investigate. Maybe other colleagues would like to use some of them in similar seminars for pre-service or in-service mathematics teachers.

Activity 1.

Every one who has studied geometry knows that the quadrilateral formed by joining the midpoints of the sides of a convex quadrilateral is a parallelogram, the area of which is half the area of the outer quadrilateral. a) What would happen if we join trisection points? b) Investigate further to present some conjectures about the quadrilateral formed when the sides of a quadrilateral are divided into 4,5,…,n congruent parts and the first division points are joined in order. How do the ratios of the areas compare in these cases?

Activity 2.

Start with an equilateral triangle, $\triangle ABC$. Construct a new triangle, $\triangle KMN$, such that its vertices are the midpoints of the sides of $\triangle ABC$. As it is well known, the area of $\triangle KMN$ is one-fourth the area of $\triangle ABC$. What if we place each of the points K,M, and N one-third of the way, three-fourths of the way, or any other fractional part of the way from one end of their line segments to the other? How do the ratios of the areas compare in these cases?

Activity 3.

The sides of a quadrilateral have lengths a, b, c, and d. The diagonals have lengths k and m. For what kind(s) of quadrilaterals does the following formula work for: $a^2 + b^2 + c^2 + d^2 = k^2 + m^2$?

Activity 4.

In every triangle the Euler line includes the circumcenter, centroid, and the orthocenter of the triangle. Construct the Euler line for an equilateral, an isosceles, an obtuse and a scalene acute triangle. What do you observe? Can the Euler line be parallel to a side of a triangle? Investigate.

Activity5.

Construct a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ using sliders to set up the coefficients a, b, c, and d. After graphing several cubic polynomials by changing the values of a, b, c, and d

i) Try to classify the cubic polynomials by the shape of their graphs. How is this classification determined by the expression $b^2 - 3ac$?

ii) Investigate the case where the cubic graph has a relative maximum, a relative minimum and an inflection point. Compare their coordinates.

Activity 6.

Construct the parabola $f(x) = ax^2 + bx + c$ setting up sliders for the coefficients a, b, and c. Next take two points A(k,f(k)) and B(m,f(m)) and draw the secant AB as well as the tangent line of parabola at the average of the x-coordinates of the two original points. What do you observe? Repeat the experiment sketching several graphs by controlling the values of a, b, and c.

Activity 7.

Construct the parabola $f(x) = ax^2 + bx + c$ where a, b, and c are controlled by sliders and find its vertex. Investigate the loci of the vertex when b is allowed to vary while a and c remain constant. Create a sketch that traces this curve. Can you find the equation of this curve?

5. Concluding remarks

We believe that a principal concern of mathematics teaching is to enable students to experience the thrill and satisfaction of mathematics. They should be able to apply their own ideas and curiosity, together with technology developed formally in the classroom, to solve problems, test and prove results. It is possible to improve student's experience of problem solving, understanding and proof, and give them a feel for what mathematics is really about.

To achieve this, requires a better mathematics teacher, a well prepared mathematics teacher to implement the guidelines of professional organizations in order to improve the teaching and learning of mathematics. The Mathematical Association of America [13] strongly supports revisions in content and teaching methodology in traditional undergraduate mathematics courses, at both lower and upper levels stating that teachers "must have opportunities in their collegiate courses to do mathematics: explore, analyze, construct models, collect and represent data, present arguments, and solve problems".

Polya was emphatic in describing the importance of discovery and creativity for the classroom teacher. "Nobody can give away what he has not got. No teacher can impart to his students the experience of discovery if he was not got it himself …The most important thing for prospective teachers is the spirit of creative work" [10].

Unfortunately in contrast to the huge amount of problem-solving literature produced in the last 20 years, Polya's ideas on teaching training are still far from being realized on a broad basis. It will not do to expose prospective teachers only to theories on problem-oriented teaching (its conditions, its possibilities, its difficulties, etc.). What must be done is to provide such training that problem orientation and "doing mathematics with children" will become fundamental to the role of the teacher – a part of the teacher's personality and an essential element in his professional life. That means that the teacher should do mathematics. "The best way to teach teachers is to make them ask and do what they in turn will make their students ask and do" [10].

The purpose of this paper was to present some components of a seminar for in-service high school teachers of mathematics that offers the participants a totally new experience in creative work, with opportunities to formulate, communicate and support original mathematical conjectures using technology. The teaching of reasoning is enhanced by a multidimensional approach. For many students, a more hands-on, discovery approach is a necessary prerequisite to move abstract generalizations. Technology allows that concrete experience.

The mathematics teachers who participated in the seminar were pleased to return to their classrooms with a great number of experiences and ideas for immediate use. As one participant noted several months later in looking back at his experiences, "The seminar gave me valuable insights of how to use technology to explore and investigate new mathematical situations, how to become a better problem solver myself and thus a better teacher of problem-solving. I realized that inductive and deductive reasoning are used together as complementary processes to give the fullest understanding of a theorem".

Another teacher said: "We worked on open-ended problems, we used technology to make and check conjectures, we worked in a cooperative-learning setting, and we found results that were new for all of us. Needles to say, some important lessons learned were brought back to our classrooms".

As has been said many times in the past, one of the major goals of mathematics teaching is to lead students to appreciate the power and beauty of mathematical thought. The technology is available to implement it, and the time has come to change our approaches to the teaching of mathematics.

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