Lecture Notes and Exercises
on
Advanced Level Pure Mathematics

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Chapter 1

Statement Calculus

1.1 Statement

Definition 1.1. A statement is a sentence of which it is **meaningful** to say whether it is true or false.

Definition 1.2. An open sentence is a sentence which is **NOT** a statement.

Example 1.1.

1. "8 is not an integer." is a statement. Actually, it is a false statement.
2. "Please keep quiet." is an open sentence because it is meaningless to say whether it is true or false.

Exercise 1.1.

Determine whether each of the following sentences is a statement or an open sentence.

1. $3 < 2$
2. It is going to rain on the coming Sunday.
3. 2 is a prime number.
4. Ha! Ha!
5. Sum of any two even numbers is an even number.
6. There is a cup on my desk.
7. The sun is happy.
8. $1 + 1 \neq 2$.
9. You are very *@#!?.
10. 1 is a root of $x^2 + 2x - 3 = 0$.

1.2 Operations on statements

In this section, let p, q, r be statements.

Definition 1.3. $\sim p$ means the negation of p.

Example 1.2.

$p =$ John is sitting on a chair.
$\sim p =$ John is not sitting on a chair.

The effect of "$\sim$" on a statement can be best illustrated by the **truth table**.

p	$\sim p$
T	**F**
F	**T**

Exercise 1.2.

Write down the negation of the statements in the previous exercise.

Definition 1.4. $p \wedge q$ means the statement "p and q".

Example 1.3.

$p =$ John's age is greater than 10.
$q =$ John's age is smaller than 30.
$p \wedge q =$ John's age is greater than 10 and smaller than 30.

Exercise 1.3.

Fill the following truth table.

p	q	$p \wedge q$

Definition 1.5. $p \vee q$ means the statement "p or q".

Example 1.4.

$p =$ John's age is smaller than 15.
$q =$ John's age is greater than 36.
$p \vee q =$ John's age is smaller than 15 or greater than 36.

Exercise 1.4.

Fill the following truth table.

p	q	$p \vee q$

Definition 1.6. $p \to q$ means the statement "If p, then q". This is called the **conditional** of two statement p and q. p is called the **hypothesis** and q is called the **consequent**.

Example 1.5.

$p =$ Jack is a dog.
$q =$ Jack has two legs.
$p \to q =$ If Jack is a dog, then Jack has two legs.

Exercise 1.5.

Fill the following truth table.

p	q	$p \to q$

Remark 1.1.

1. Notice that if the hypothesis p is false, $p \to q$ is true whenever q is true or false (Why?).
2. If $p \to q$ is **always** true, we usually write $p \to q$ as $p \Rightarrow q$.
3. There are three other common ways to express $p \Rightarrow q$:
   (a) "p **implies** q".
   (b) "p is a **sufficient condition** for q".
   (c) "q is a **necessary condition** for p".
4. When proving $p \Rightarrow q$, it is enough to start with the assumption that p is true, and then show that q must also be true.
5. It is not hard to see that if $p \Rightarrow q$ and $q \Rightarrow r$, then $p \Rightarrow r$ (Can you prove it?).
6. A statement which is always true is called a **tautology**. A statement which is always false is called a **contradition**.

Example 1.6. Prove that n and m are even numbers implies $n + m$ is also an even number.

Solution. n and m are even numbers
$\Rightarrow n = 2k$, $m = 2l$ for some integers k and l
$\Rightarrow n + m = 2k + 2l = 2(k + l)$
$\Rightarrow n + m$ is divisible by 2
$\Rightarrow n + m$ is an even number.

Definition 1.7. $p \leftrightarrow q$ means "If p, then q. And also if q, then p" i.e. $(p \to q) \wedge (q \to p)$. This is called the **biconditional** of two statements p and q.

Exercise 1.6.

Fill the following truth table.

p	q	$p \leftrightarrow q$

Remark 1.2.

1. If $p \leftrightarrow q$ is **always** true, we usually write $p \to q$ as $p \Leftrightarrow q$.
2. There are four other common ways to express $p \Leftrightarrow q$:
   (a) "p **if and only if** q".
   (b) "p **iff** q" ("iff" is the abbreviation for "if and only if").
   (c) "p is a **necessary and sufficient condition** for q".
   (d) "p is **equivalent** to q".
3. When proving $p \Leftrightarrow q$, you need to show both $p \Rightarrow q$ ("Only if" part) and $q \Rightarrow p$ ("If" part).

Definition 1.8. p and q are equivalent if they have the same truth value under the same conditions i.e. they have the same truth table. It is denoted by $p \equiv q$.

Example 1.7. Show that $p \to q \equiv (\sim p) \vee q$.

Solution. Consider the following truth tables:

p	q	$p \leftrightarrow q$

p	q	$(\sim p) \vee q$

Since their truth tables are the same, $p \to q \equiv (\sim p) \vee q$.

Exercise 1.7.

1. Let x, y, z be any real numbers. Show that $x^2 + y^2 + z^2 = 0 \Leftrightarrow x = 0, y = 0, z = 0$.
2. Prove "De Morgan's Laws":
   (a) $\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$
   (b) $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$

   Hence write down the negation of the following statements:

   (a) Susan passes her Math test and Bill passes his Chinese test.
   (b) Peter goes to school by bus or by MTR.
3. (a) Show that $p \rightarrow q \equiv (\sim q) \rightarrow (\sim p)$.
   (b) Let m be a positive integer. Use (a) to prove that if m^2 is odd, then m is also odd (Note: This technique of proof is called the "method of contradition").

1.3 Quantifiers

Definition 1.9. "$\forall$" means "for all" and is called the **Universal quantifier**.

Definition 1.10. "$\exists$" means "there exists" (or "for some") and is called the **Existential quantifier**.

Example 1.8. Usage of quantifiers:

1. $\forall$ real number x, $x^2 \geq 0$.
2. $\exists$ positive integers x, y, z such that $x^2 + y^2 = z^2$.

Exercise 1.8.

Write down the negation of the above examples.

Remark 1.3.

In general, let $p(x)$ be a statement about x. we have:

1. $\sim [\forall x, p(x)] \Leftrightarrow \exists x, \sim p(x)$
2. $\sim [\exists x, p(x)] \Leftrightarrow \forall x, \sim p(x)$

Example 1.9. Disprove the statement "$\forall$ prime number $p > 2$, $p + 26$ is also a prime number".

Solution. To disprove this statement, we need to show that its negation is true
i.e. $\exists p > 2$ such that $p + 26$ is NOT a prime number.

Consider $p = 7$, $7 + 26 = 33$. Obviously, 33 is not a prime number.
$\therefore$ The given statement is disproved.
(Note: This method is called "disprove by counter-example".)

Chapter 2

Mathematical Induction

2.1 First principle of mathematical induction

Let us recall the **first principle of mathematical induction**:

Theorem 2.1. Let $P(n)$ be a statement about integer n. $P(n)$ is true for all integer $n \geq n_0$ if and only if the following two conditions are satisfied:

1. $P(n_0)$ is true,
2. If $P(k)$ is true for $k \geq n_0$, then $P(k+1)$ is also true.

Example 2.1. (Proving identities) Prove that for any integer $n \geq 1$,

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3).$$

Solution.

Exercise 2.1.

1. Prove that $(1-\frac{1}{2^2})(1-\frac{1}{3^2})(1-\frac{1}{4^2})\cdots(1-\frac{1}{n^2}) = \frac{n+1}{2n}$ for any integer $n \geq 2$.
2. Prove that $\sum_{k=1}^{n} k(k!) = (k+1)! - 1$ for any integer $n \geq 1$.

Example 2.2. (Proving inequalities) Prove that $2^n > n^2$ for any integer $n \geq 5$.

Solution.

Exercise 2.2.

1. Prove that $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}$ for any integer $n > 1$.
2. Prove that $2^n > n^3$ for any integer $n \geq 10$.

Example 2.3. (Proving divisibility) Prove that $3^{4n+2} + 2^{6n+3}$ is divisible by 17 for any integer $n \geq 1$.

Solution.

Exercise 2.3.

1. Prove that $6^n + 4$ is divisible by 5 for any $n \geq 1$.
2. Prove that $2^{5n} - 2^n$ is divisible by 10 for any $n \geq 1$.

Example 2.4. (Proving surd expression) Prove that for any positive integer n, there exists **unique** positive integers a_n and b_n such that

$$(1+\sqrt{5})^n = a_n + b_n\sqrt{5}$$

Solution.

Exercise 2.4.

1. For any integer $n \geq 1$, prove that there exists unique positive integers p_n, q_n such that $(\sqrt{5}+\sqrt{2})^{2n} = p_n + q_n\sqrt{10}$ and $(\sqrt{5}-\sqrt{2})^{2n} = p_n - q_n\sqrt{10}$.

If we want to prove a statement holds for all positive odd (even) integers, we need to modify the first principle of mathematical induction:

Theorem 2.2. Let $P(n)$ be a statement about integer n. $P(n)$ is true for all odd (even) integer $n \geq n_0$ if and only if the following two conditions are satisfied:

1. $P(n_0)$ is true,
2. If $P(k)$ is true for $k \geq n_0$, then $P(k+2)$ is also true.

Example 2.5. Prove, by mathematical induction, that $5^n - 3^n - 2^n$ is divisible by 30 for all positive odd integers n greater than 1.

Solution.

2.2 Second principle of mathematical induction

In the first principle of mathematical induction, the second condition can be replaced by a weaker one:

Theorem 2.3. Let $P(n)$ be a statement about integer n. $P(n)$ is true for all integer $n \geq n_0$ if and only if the following two conditions are satisfied:

1. $P(n_0)$ is true,
2. If $P(l)$ is true for all l such that $k \geq l \geq n_0$, then $P(k+1)$ is also true.

This is called the **second principle of mathematical induction**.

The following is called the **mathematical induction with double assumptions**, which is a special case of the second principle of mathematical induction.

Theorem 2.4. Let $P(n)$ be a statement about integer n. $P(n)$ is true for all integer $n \geq n_0$ if and only if the following two conditions are satisfied:

1. $P(n_0)$ and $P(n_0+1)$ is true,
2. If $P(k)$ and $P(k+1)$ is true for $k \geq n_0$, then $P(k+2)$ is also true.

Example 2.6. (Double assumptions) Let $a_1 = 1$, $a_2 = 3$ and $a_n = a_{n-2} + a_{n-1}$ for all $n \geq 3$. Prove that

$$a_n = \alpha^n + \beta^n$$

for $n \geq 1$, where α, β are roots of the equation $x^2 - x - 1 = 0$.

Solution.

Example 2.7. (Double assumptions) Let n be a postive integer. Show, by induction, that

$$\frac{1}{\sqrt{5}}\left((1+\sqrt{5})^n-(1-\sqrt{5})^n\right) \text{ is divisible by } 2^n.$$

Solution.

Example 2.8. (Triple assumptions) Let $\{a_n\}$ be a sequence of real numbers, where

$$a_0=1, a_1=6, a_2=45 \quad \text{and} \quad a_n-a_{n+1}+\frac{1}{3}a_{n+2}-\frac{1}{27}a_{n+3}=0$$

for $n=0,1,2,\ldots$. Using mathematical induction, show that

$$a_n=3^n(n^2+1) \quad \text{for} \quad n=0,1,2,\ldots.$$

Solution.

Example 2.9. (More assumptions) Let $\{a_n\}$ be a sequence of real numbers, where

$$a_1 = 2 \quad \text{and} \quad a_1 + a_2 + \cdots + a_n = n^2 + n \quad \text{for all} \quad n = 1, 2, \ldots$$

Show that for all postive integers n, $a_n = 2n$.
Solution.

Exercise 2.5.

1. Suppose a sequence $\{a_n\}$ satisfies the condition: $a_0 = 0, a_1 = -7$ and $a_{n+2} + a_{n+1} - 12a_n = 0$ for $n = 0, 1, 2.\ldots.$ Prove that $a_n = -3^n + (-4)^n$ for $n = 0, 1, 2.\ldots.$

2. Suppose a sequence $\{a_n\}$ satisfies the condition: $a_0 = 1, a_1 = \dfrac{1}{2}$ and $2(n+2)a_{n+2} - 3na_{n+1} + (n-1)a_n = 0$. Prove that $a_n = \dfrac{1}{2^n}$ for all integer $n \geq 1$.

3. Let $\{a_n\}$ be a sequence of non-negative integers such that
$$n \leq \sum_{k=1}^{n} a_k^2 \leq n + 1 + (-1)^n$$
for $n = 1, 2, 3, \ldots.$ Prove that $a_n = 1$ for $n \geq 1$.

4. Let $\{a_n\}$ be a sequence of positive numbers such that
$$a_1 + a_2 + \cdots + a_n = \left(\frac{1 + a_n}{2}\right)^2$$
for $n = 1, 2, 3, \ldots.$ Prove by induction that $a_n = 2n - 1$ for $n = 1, 2, 3, \ldots.$

Chapter 3

Polynomials and Rational Functions

3.1 Polynomials

3.1.1 Basic concepts

Definition 3.1. A **polynomial in** x is an algebraic expression of the following form:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_1, a_2, \ldots, a_n$ are constants and n is a non-negative integer.

1. $a_1, a_2, \ldots, a_n$ are called the **coefficients of the polynomial**.
2. If $a_n \neq 0$, a_n is the **leading coefficient of the polynomial**.
3. The polynomial is a **zero polynomial** if $a_0 = a_1 = \cdots = a_n = 0$.
4. The polynomial is a **constant polynomial** if $a_1 = \cdots = a_n = 0$.
5. The polynomial is a **monic polynomial** if $a_n = 1$.

Definition 3.2. Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and $q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$ be two polynomials in x. We say that they are **equal** if $m = n$ and $a_i = b_i$ for $i = 0, 1, \ldots, n$. In this case, we write $p(x) \equiv q(x)$ (It means $p(x) = q(x)$ for all x).

Example 3.1. Find the values of a, b, c and d such that

$$x^3 \equiv a(x-1)(x-2)(x-3) + b(x-1)(x-2) + c(x-1) + d$$

Solution.

Definition 3.3. Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial and $a_0 \neq 0$. Then n is the **degree of** $p(x)$. We write $\deg p(x) = n$.

Theorem 3.1. Suppose $p(x), q(x)$ are two polynomials. Then we have

1. $\deg(p(x) + q(x)) \leq \max\{\deg p(x), \deg q(x)\}$
2. $\deg(p(x)q(x)) = \deg p(x) + \deg q(x)$

Remark 3.1.

1. $\deg p(x) = 0$ if $p(x)$ is any non-zero constant polynomial.
2. $\deg p(x) = -\infty$ if $p(x)$ is a zero polynomial.

Example 3.2. Let $p(x) = 2x^2 - x + 1$ and $q(x) = x^3 - x^2 + 5x + 8$. Find $\deg p(x)$, $\deg q(x)$, $\deg(p(x) + q(x))$ and $\deg(p(x)q(x))$. Hence verify the above theorem.
Solution.

Example 3.3. Let $f(x)$, $g(x)$ and $h(x)$ be three non-zero polynomials such that $f(x) = h(x)g(x)$. Show that $\deg f(x) > \max\{\deg g(x), \deg h(x)\}$.
Solution.

Exercise 3.1.

1. Find an example of $p(x), q(x)$ such that $\deg(p(x) + q(x)) < \max\{\deg p(x), \deg q(x)\}$.
2. Suppose $p(x)$ is a polynomial with degree $= n$. Find $\deg(p(x))^3$ and $\deg(p(p(p(x))))$.

Theorem 3.2. (Division algorithm) Given any two polynomials $f(x), g(x)$ with $g(x) \neq 0$. There exists **unique** $q(x)$ and $r(x)$ such that the following two conditions are satisfied:

1. $f(x) = g(x)q(x) + r(x)$ and
2. $\deg r(x) < \deg g(x)$.

$r(x)$ and $q(x)$ are called the **remainder** and the **quotient** respectively.
If $r(x) = 0$, $p(x)$ is said to be **divisible** by $q(x)$. In other words, $q(x)$ is a **factor** of $p(x)$.

Example 3.4. Find the quotient and remainder when $x^4 - 2x^3 - 2x^2 + x - 3$ is divided by $2x^2 - 1$.
Solution.

Example 3.5. (Synthetic division) Find the quotient and remainder when $4x^4 - 3x^3 - 2x^2 - 5x - 1$ is divided by $x - 3$.
Solution.

Theorem 3.3. (Remainder Theoerm) Suppose a is a real number and $f(x)$ is a polynomial such that $f(a) = r$. Then $f(x)$ has the remainder r when divided by $(x - a)$. In particular, if $r = 0$, $f(x)$ is divisible by $(x - a)$.

Example 3.6. A polynomial $f(x)$ is divisible by $(x - 1)$. The remainders when $f(x)$ is divided by $(x - 2)$ and $(x - 3)$ are -7 and -20 respectively. When $f(x)$ is divided by $(x - 1)(x - 2)(x - 3)$, the remainder is $ax^2 + bx + c$, where a, b and c are constants. Find the values of a, b and c.

Solution.

Exercise 3.2.

1. A polynomial gives remainder $2x + 5$ when divided by $(x - 1)(x + 2)$. Find the remainders when it is divided by $x - 1$ and $x + 2$ separately.

2. A polynomial gives remainder 2 and 1 when divided by $x + 1$ and $x - 4$ respectively. Find the remainder when its is divided by $(x + 1)(x - 4)$.

3. A cubic polynomial gives remainders $5x - 7, 12x - 1$ when divided by $x^2 - x + 2$ and $x^2 + x - 1$ respectively. Find the polynomial.

4. Let $f(x) = x^4 + px^2 + qx + a^2$. If $f(x)$ is divisible by $x^2 - 1$, prove that $f(x)$ is also divisible by $x^2 - a^2$.

3.1.2 Euclidean Algorithm

Definition 3.4. Let $f(x)$ and $g(x)$ be two non-zero polynomials. If there exists a polynomial $h(x)$ such that

$$f(x) = g(x)h(x)$$

$g(x)$ is called a **factor** of $f(x)$.

Theorem 3.4. If $g(x)$ is a factor of a non-zero polynomial $f(x)$, then

$$\deg g(x) \leq \deg f(x)$$

Definition 3.5. Let $f(x)$ and $g(x)$ be two non-zero polynomials.

1. $h(x)$ is a **common factor(common divisor)** of $f(x)$ and $g(x)$ iff $h(x)$ is a factor of both $f(x)$ and $g(x)$.
2. $h(x)$ is the **greatest common divisor *gcd* (highest common factor *hcf*)** of $f(x)$ and $g(x)$ iff $h(x)$ is the common factor of both $f(x)$ and $g(x)$ with the highest degree. We write $h(x) = \gcd(f(x), h(x))$.
3. $f(x)$ and $g(x)$ are **relatively prime** if their gcd is constant.

Remark 3.2.

1. Zero polynomial can *never* be a factor.
2. Every non-zero polynomial is a factor of zero polynomial.
3. Every common factor of $f(x)$ and $g(x)$ is a factor of $\gcd(f(x), g(x))$.
4. $\gcd(f(x), g(x))$ is *unique* up to constant multiplication i.e if both $h(x)$ and $k(x)$ are gcd, $h(x) = ck(x)$ for some constant c.

Example 3.7. (Euclidean Algorithm)

1. Find $\gcd(f(x), g(x))$ where $f(x) = 4x^4 - 2x^3 - 16x^2 + 5x + 9$ and $g(x) = 2x^3 - x^2 - 5x + 4$.
2. Find polynomials $p(x)$ and $q(x)$ such that

$$\gcd(f(x), g(x)) = p(x)f(x) + q(x)g(x)$$

Solution.

Exercise 3.3.

1. Let $f(x) = x^3+x^2+x-3$ and $g(x) = x^2-1$. Find $\gcd(f(x), g(x))$. Moreover, find polynomials $a(x), b(x)$ such that $\gcd(f(x), g(x)) = a(x)f(x) + b(x)g(x)$.
2. (a) Show that $x^2 + 1$ and $x^3 + x^2 + 1$ are relatively prime and find polynomials $m(x)$ and $n(x)$ such that
$$(x^3 + x^2 + 1)m(x) + (x^2 + 1)n(x) \equiv 1$$
   (b) Find a polynomial $h(x)$ such that $h(x)$ is divisible by x^2+1 and $h(x)+1$ is divisible by $x^3 + x^2 + 1$.

3.1.3 Roots of polynomials

Definition 3.6. Let $f(x)$ be a polynomial. c is a **root(zero)** of $f(x)$ if $f(c) = 0$.

Remark 3.3.

1. Root of a polynomial can be a complex number.
2. Any non-zero constant polynomial has no roots.
3. Every numbers is a root of a zero polynomial.

Theorem 3.5. (Factor Theorem) Let $f(x)$ be a polynomial. Then

$$c \text{ is a root} \quad \Leftrightarrow \quad (x-c) \text{ is a factor of } f(x)$$

Proof.

The following is the most important theorem concerning roots of polynomials. It is called the **Fundamental Theorem of Algebra**.

Theorem 3.6. (Fundamental Theorem of Algebra) For any polynomial of degree ≥ 1, there exists at least one root.

The direct consequence of the Fundamental Theorem of Algebra is as follows:

Theorem 3.7. For any polynomial of degree ≥ 1, there exists exactly n roots. Moreover, suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has roots $r_1, r_2, \ldots, r_n$. We have

$$f(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$$

Proof.

The following is an important application of the above theorem.

Theorem 3.8. Let $f(x)$ be a polynomial with $\deg f(x) \leq n$, where $n > 0$. If there exists more than n distinct roots of $f(x)$, then $f(x)$ is a zero polynomial.

Proof.

Example 3.8.

(a) Prove the identity

$$\frac{a^2(x-b)(x-c)}{(a-b)(a-c)}+\frac{b^2(x-c)(x-a)}{(b-c)(b-a)}+\frac{c^2(x-a)(x-b)}{(c-a)(c-b)}\equiv x^2$$

where a, b, c are distinct real numbers.

(b) Using (a), show that

$$\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}=0$$

for any distinct real numbers a, b, c.

Solution.

Exercise 3.4.

1. (a) Show that for any real numbers a, b and c,

   $$a^3(b-c)+b^3(c-a)+c^3(a-b)=-(a+b+c)(a-b)(b-c)(c-a)$$

   (b) Hence, or otherwise, show that

   $$\frac{(x+a)^3}{(a-b)(a-c)}+\frac{(x+b)^3}{(b-a)(b-c)}+\frac{(x+c)^3}{(c-a)(c-b)}\equiv 3x+a+b+c$$

   where a, b, c are distinct non-zero real numbers.

2. (a) Let $f(x)$ be a quadratic polynomial such that $f(a)=\alpha$, $f(b)=\beta$ and $f(c)=\gamma$, where a, b and c are distinct real numbers, show that

   $$f(x)\equiv\frac{\alpha(x-b)(x-c)}{(a-b)(a-c)}+\frac{\beta(x-a)(x-c)}{(b-a)(b-c)}+\frac{\gamma(x-a)(x-b)}{(c-a)(c-b)}$$

   (b) Hence show that every quadratic polynomial can be written in the form

   $$A(x-b)(x-c)+B(x-a)(x-c)+C(x-a)(x-b)$$

   where A, B and C are real numbers.

3.1.4 Mulitple roots

Definition 3.7. If a polynomial $f(x)$ has root α such that $f(x)$ is divisible by $(x-\alpha)^k$ but not $(x-\alpha)^{k+1}$, where k is a positive integer (in other words, $f(x) = (x-\alpha)^k g(x)$ for some polynomial $g(x)$ such that $g(\alpha) \neq 0$), then α is called a root of $f(x)$ of **multiplicity** k .

Definition 3.8. Let $f(x)$ be a non-zero polynomial.

1. α is a **simple root** of $f(x)$ if α is a root of $f(x)$ of multiplicity 1 .
2. If α is a root of $f(x)$ whose multiplicity is greater than 1, then it is called a **multiple (repeated) root** of $f(x)$.
3. In particular, if α is a root of $f(x)$ of multiplicity 2, then it is called a **double root** of $f(x)$.

Theorem 3.9. Let $f(x)$ be a polynomial. α is a root of $f(x)$ of multplicity k ($k > 1$) iff α is a root of $f'(x)$ of multiplicity $k-1$ and $f(\alpha) = 0$.

Proof.

In particular, we have the simple version of the above theorem:

Theorem 3.10. If α is a multiple root of $f(x)$, then $f(\alpha) = f'(\alpha) = 0$.

Example 3.9. Show that the equation $3x^4 - 8x^3 - 6x^2 + 24x + 1 = 0$ cannot have a double root.

Solution.

Example 3.10. Suppose $f(x) = x^3 + px + q$, where p and q are real numbers. Show that if $f(x) = 0$ has a multiple root, then $27p^2 + 4p^3 = 0$.

Solution.

Exercise 3.5.

1. If the equation $x^3 + ax^2 + bx + c = 0$ has a multiple root, show that this roots is $\dfrac{9c - ab}{2(a^2 - 3b)}$ where $3b \neq a^2$.

2. $f(x)$ and $g(x)$ are given polynomials which are relatively prime. Prove that the values of k for which the equation
$$f(x) - kg(x) = 0$$
has a multiple root are given by $\dfrac{f(\alpha)}{g(\alpha)}$, where α is a root of the equation
$$f(x)g'(x) - f'(x)g(x) = 0$$
Hence, or otherwise, find the values of k for which the equation $x^3 - 3x^2 + 3kx - 1 = 0$ has a multiple root. Solve the equation for each case.

3. If α is a double root of the equation $x^5 + 5qx^3 + 5rx^2 + t = 0$, prove that α is also a root of
$$3r^2 - 6q^2x - 4qr + t = 0.$$

4. Solve the equation $x^4 - 11x^3 + 44x^2 - 76x + 48 = 0$, given that it has a multiple root.

3.1.5 Rational roots

Definition 3.9. Let $f(x)$ be a polynomial. If α is a root of $f(x)$ and it is a rational number, then α is called a **rational root** of $f(x)$.

The following is the main theorem about rational roots:

Theorem 3.11. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficents and r, s be two relatively prime integers i.e. $\gcd(r, s) = 1$. If $\frac{s}{r}$ is a root of $f(x) = 0$, then s divides a_0 and r divides a_n.

Proof.

Example 3.11. Show that it is impossible to factorize $2x^4 + 5x + 1$ into factors with rational coefficients.

Solution.

Example 3.12. Let p be a prime number and α be a real number. If α is a root of the equation $x^3 - px + p = 0$, prove that α is a irrational number.

Solution.

Exercise 3.6.

1. Show that $\sqrt{3} - \sqrt{2}$ satisfies the equation $x^4 - 10x + 1 = 0$. Hence deduce that $\sqrt{3} - \sqrt{2}$ is an irrational number.

2. Let a be an integer and $p(x)$ be the polynomial

$$2x^{n+2} - 5x^{n+1} + 2x^n - 2ax^3 + (5a+2)x^2 - (2a+5)x + 2$$

where $n > 1$. Show that $p(2) = 0$. Moreover, show that $p(x) = 0$ has exactly two rational roots provided that a does not take one of values $0, 2, -2$.

3.1.6 Relations between roots and coefficients

Given a quadratic polynomial $f(x) = a_2x^2 + a_1x + a_0$. By the Fundamental Theorem of Algebra, there exists α and β be roots of $f(x) = 0$ such that

$$f(x) = a_2(x-\alpha)(x-\beta)$$

By expansion, we get

$$a_2x^2 + a_1x + a_0 = a_2x^2 - a_2(\alpha+\beta)x + a_2\alpha\beta$$

Therefore, we have the following results by comparing coefficients:

1. $\alpha + \beta = -\dfrac{a_1}{a_2}$

2. $\alpha\beta = \dfrac{a_0}{a_2}$

Given a cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. Let α, β and γ be roots of $f(x) = 0$. Similarly, we can write

$$f(x) = a_3(x - \alpha)(x - \beta)(x - \gamma)$$

By expansion, we get

$$a_3x^3 + a_2x^2 + a_1x + a_0 =$$

Therefore, we have the following results by comparing coefficients:

1.

2.

3.

In general, suppose $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are roots of $f(x) = 0$. We have the following relations between roots and coefficients:

Example 3.13.

1. Suppose the cubic equation $x^3 + px^2 + qx + r = 0$ where p, q and r are real numbers, has three real roots. Prove that the three roots form an arithmetic sequence iff $\dfrac{-p}{3}$ is a root of the equation.

2. Find the two values of p such that the equation $x^3 + px^2 + 21x + p = 0$ has three real roots that form an arithmetic sequence.

Solution.

Example 3.14. Suppose the equation $x^3 + px^2 + qx + 1 = 0$ has three real roots.

1. If the roots of the equation can form a geometric sequence, show that $p = q$.
2. If $p = q$, show that -1 is a root of the equation and the three roots of the equation can form a geometric sequence.

Solution.

Example 3.15. Let α, β and γ be roots of the equation $x^3 + px^2 + qx + r = 0$.

1. Express $\alpha^2 + \beta^2 + \gamma^2$, $\alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2$ and $\alpha^2\beta^2\gamma^2$ in terms of p, q and r.
2. Hence find a cubic equation whose roots are squares of the roots $x^3 - 4x + 1 = 0$.

Solution.

Here are some useful algebraic identities:

1. $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\alpha\gamma)$
2. $\alpha^3+\beta^3+\gamma^3 = 3\alpha\beta\gamma+(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-(\alpha\beta+\beta\gamma+\alpha\gamma))$

Exercise 3.7.

1. Suppose α, β, γ be three roots of $2x^3+x^2-x+1=0$.
   (a) Find $\alpha^2+\beta^2+\gamma^2$.
   (b) Find $\alpha^2\beta^2+\beta^2\gamma^2+\alpha^2\gamma^2$.
   (c) Find a polynomial with roots $\alpha^2, \beta^2, \gamma^2$.
2. (a) Prove that the roots of $x^3+3ax^2+3bx+c=0$ are in A.P. if and only if $2a^3-3ab+c=0$.
   (b) Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of $x^3+4x^2+2x+k=0$, where k is a constant. Using (i), or otherwise, show that there are exactly two values of k for which $\alpha^2, \beta^2, \gamma^2$ are in A.P..
3. (a) If α, β, γ are the roots of $x^3+qx+r=0$, find the equation whose roots are $(\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta)^2$.
   (b) If α, β, γ are the roots of $x^3+px^2+qx+r=0$, express $(\alpha^2-\beta\gamma)(\beta^2-\gamma\alpha)(\gamma^2-\alpha\beta)$ in terms of p, q, r.

3.1.7 Transformation of polynomials

By transformation of the variable, we may be able to simplify the polynomial equation and hence find its roots.

Example 3.16. By using the transformation $y = x+1$, solve the equation

$$x^4-4x^3+x^2+6x+2=0$$

Solution.

Example 3.17. By using the transformation $t = x + \frac{1}{x}$, find two real roots of the equation $6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0$.

Solution.

Example 3.18. Suppose α, β and γ are roots of the equation $x^3 + 2x^2 - 2 = 0$.

(a) Show that α, β and γ cannot be equal to 0.

(b) By using the method of tansformation, find a equation whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

Solution.

Example 3.19.

(a) Show that $(a+b)$ is a root of the equation $x^3 - 3abx - (a^3 + b^3) = 0$.

(b) By expressing $f(x) = x^3 - 6x - 6 = 0$ into the form of the above equation, find a real root of $f(x) = 0$.

(c) By putting $x = y + k$, transform the equation $g(x) = x^3 + 3x^2 - 3x - 11 = 0$ into the form $y^3 + py + q = 0$. Hence, find a real root of $g(x) = 0$.

Solution.

Exercise 3.8.

1. Given that α, β, γ are roots of $x^3 + x^2 + 3 = 0$. By considering the transformation $y = \dfrac{1}{1+x}$, find a polynomial whose roots are $\dfrac{1}{1+\alpha}, \dfrac{1}{1+\beta}, \dfrac{1}{1+\gamma}$.

2. (a) Let $y = x + \dfrac{1}{x}$ and $v_r = x^r + \dfrac{1}{x^r}$. Prove that $v_{r+1} = yv_r - v_{r-1}$. Hence express $x^2 + \dfrac{1}{x^2}$, $x^3 + \dfrac{1}{x^3}$, $x^4 + \dfrac{1}{x^4}$ in terms of y.

   (b) By using the transformation $y = x + \dfrac{1}{x}$ or any other method, solve the equation $x^{10} - 3x^8 + 5x^6 - 5x^4 + 3x^2 - 1 = 0$.

3. Let α and β be the roots of the quadratic equation $x^2 + 2px - q^3 = 0$, where p and q are real constants and $q > 0$.

   (a) Show that α and β are distinct real numbers.

   (b) Express $\alpha + \beta$ and $\alpha\beta$ in terms of p and q.

   (c) Show that $\alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}}$ is a root of the cubic equation $x^3 + 3qx + 2p = 0$.

   (d) By putting $y = x - 1$, find a real root of the equation $x^3 - 3x^2 + 9x - 9 = 0$.

3.2 Rational functions and Parital fractions

Definition 3.10. **A rational function** is the ratio of two polynomials i.e. it can be written as $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials ($q(x)$ cannot be zero polynomial).

Moreover, it is said to be a **proper fraction** if $\deg p(x) < \deg q(x)$. Otherwise, it is a **improper fraction**.

3.2.1 An overview

Given a rational function $f(x) = \frac{p(x)}{q(x)}$. Suppose $q(x)$ can be factorized into factors with real coefficients. Then $f(x)$ can be expressed as sum or difference of simpler fractions. These simpler fractions are called **partial fractions**. The general procedure for writing a rational function into partial fractions is as follows:

1. Is $f(x)$ a proper fraction?
   (Yes) Go to next step.
   (No) Use long division to rewrite $f(x)$ as the sum of a polynomial and a proper fraction.
2. Factorize $q(x)$ completely into factors with real coefficients.
3. Express $f(x)$ as partial fractions with unknowns according to the rules stated in the next section.
4. Determine the unknowns.
5. Write $f(x)$ in partial fractions.

Example 3.20. Express $\frac{x^3+2x-1}{x^2-1}$ in partial fractions.

1. It is an improper fraction. Therefore, by long division, we obtain the following:
$$\begin{aligned} x^3+2x-1 &= x(x^2-1)+3x-1 \\ \therefore \frac{x^3+2x-1}{x^2-1} &= x+\frac{3x-1}{x^2-1} \end{aligned}$$
2. Consider $\frac{3x-1}{x^2-1}$. Factorize the denominator completely, we get
$$\frac{3x-1}{x^2-1} = \frac{3x-1}{(x+1)(x-1)}.$$
3. Write $\frac{3x-1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$, where A, B are constants to be determined.
4. By calculation, we have $A = 1$, $B = 2$.
5. Hence $\frac{x^3+2x-1}{x^2-1} = x + \frac{1}{x+1} + \frac{2}{x-1}$.

3.2.2 Partial fractions

When expressing a rational function in partial fractions, different types of factors in the denominators correspond to different terms in partial fractions. There are four types:

Type I - Non-repeated linear factor $(ax+b)$ corresponds to the fraction $\dfrac{A}{ax+b}$, where A is a constant.

Example 3.21. Express $\dfrac{2x}{(x+2)(x-2)}$ in partial fractions.

Solution.

Type II - Non-repeated quadratic factors (ax^2+bx+c) corresponds to the fraction $\dfrac{Ax+B}{ax^2+bx+c}$, where A, B are constants.

Example 3.22. Express $\dfrac{3x^2-3x+2}{(x^2+1)(x-1)}$ in partial fractions.

Solution.

Type III - Repeated linear factor $(ax+b)^k$ corresponds to the fraction

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

where $A_1, A_2, \ldots, A_k$ are constants.

Example 3.23. Express $\dfrac{x^2+x-1}{(x-1)^2x}$ in partial fractions.

Solution.

Type IV - Repeated quadratic factor $(ax^2+bx+c)^k$ corresponds to the fraction

$$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

where $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_k$ are constants.

Example 3.24. Express $\dfrac{2x^4+4x^3+3x^2+2x-2}{(x-1)(x^2+x+1)^2}$ in partial fractions.

Solution.

Example 3.25.

(a) Resolve $\dfrac{1}{k(k+1)}$ into partial fractions.

(b) Hence, express $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{99 \cdot 100}$ as a rational number.

Solution.

Example 3.26.

(a) Resolve $\dfrac{1}{k(k+2)}$ into partial fractions.

(b) Hence evaluate $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+2)}$.

Solution.

Example 3.27.

(a) Resolve $F(x) = \dfrac{2}{(x+a)(x+a+1)(x+a+2)}$ into partial fractions, where a is a real number.

(b) Prove that $\displaystyle\sum_{k=1}^{n} F(k) = \frac{1}{1+a} - \frac{1}{2+a} - \frac{1}{n+a+1} + \frac{1}{n+a+2}$.

Solution.

Exercise 3.9.

1. Resolve the following into partial fractions:

   (a) $\dfrac{x^3+7x^2+15x+11}{(x+1)(x+2)(x+3)}$

   (b) $\dfrac{3x^2+x-8}{(x+1)^2(x-5)}$

   (c) $\dfrac{1}{(x^2+x+1)(x-1)}$

   (d) $\dfrac{x^4+x^3+2x^2+2}{(x^2+1)^2(x+1)}$

2. (a) Resolve $\dfrac{1}{x(x+3)}$ into partial fractions.

   (b) Hence evaluate $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+3)}$.

3. Let $g(x)$ be a quadratic polynomial and a, b, c are distinct real constants.

   (a) Consider the following partial fractions:

   $$\frac{g(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

   Express A, B, C in terms of a, b, c, $g(a)$, $g(b)$ and $g(c)$ only.

   (b) Hence resolve $\dfrac{x^2+1}{(x+1)(x+2)(x+3)}$ into partial fractions.

Chapter 4

Binomial Theorem

4.1 Permutations and Combinations

Definition 4.1. For any positive integer n,

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$$

Moreover, $0! = 1$.

Definition 4.2. P_r^n = number of ways to choose r objects from n different objects ($1 \leq r \leq n$) and then arrange them in order.

Theorem 4.1. For $1 \leq r \leq n$,

$$P_r^n = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Proof.

Definition 4.3. C_r^n = number of ways to choose r objects from n different objects ($r \leq n$) without regarding the order of arrangement.

Theorem 4.2. For $1 \leq r \leq n$,

$$C_r^n = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$

Proof.

Remark 4.1.

1. P_r^n is called the **permutations** of n different objects taken r at a time.
2. C_r^n is called the **combinations** of n different objects taken r at a time.
3. $P_0^n = \dfrac{n!}{(n-0)!} = 1.$
4. $C_0^n = \dfrac{n!}{0!(n-0)!} = 1.$

4.2 Binomial theorem and identities

Here are three useful identities:

Theorem 4.3. For any positive integer n and r be non-negative integer such that $r \leq n$,

1. $C_r^n = C_{n-r}^n$
2. $C_{r-1}^n + C_r^n = C_r^{n+1}$ $(r \geq 1)$
3. $C_{r-1}^{n-1} = \frac{r}{n} C_r^n$

Proof.

Remark 4.2.

1. The first identity means that the number of ways to select r objects from n objects is equal to the number of ways to select $n - r$ objects from n objects.
2. The second identity is closely related to the construction of the **Pascal Triangle**.

Theorem 4.4. (Binomial Theorm) For any non-negative integer n, we have

$$(1+x)^n = \sum_{r=0}^{n} C_r^n x^r$$

Proof.

The following is another version of Binomial Theorem.

Theorem 4.5. $(a+b)^n = \sum_{r=0}^{n} C_r^n a^r b^{n-r}$

Proof.

Example 4.1. Find the constant term in the expansion of $\left(4x - \frac{1}{2\sqrt[3]{x}}\right)^{20}$.

Solution.

Example 4.2. If the coefficient of three consecutive terms in the expansion of $(1+x)^n$ are 120, 210 and 252 respectively, find the value of n.

Solution.

Example 4.3. Prove that $\sum_{r=0}^{n-1} C_{r+1}^n x^r = \frac{1}{x}[(1+x)^n - 1]$.

Solution.

Example 4.4. Prove that

(a) $\sum_{r=0}^{n} C_r^n = 2^n$.

(b) $\sum_{r=0}^{n} (-1)^n C_r^n = 0$.

(c) $\sum_{r=0}^{n} C_{2r}^{2n} = \sum_{r=0}^{n-1} C_{2r+1}^{2n} = 2^{2n-1}$.

Solution.

Exercise 4.1.

1. Show that for one value of r the coefficient of x^r in the expansion of $(3+2x-x^2)(1+x)^{34}$ is zero.

2. Prove that $(3n+1)\left[C_n^{2n}\right]^2 = (n+1)\left[(C_n^{2n+1})^2 - (C_{n-1}^{2n})^2\right]$.

3. For $n \geq 5$, prove that $C_n^{2n} < 2^{2n-2}$.

4. Show that $\displaystyle\sum_{r=n+1}^{2n+1} C_r^{2n+1} = 2^{2n}$.

5. (a) Prove that for any positive integer n, $\displaystyle\left(1+\frac{1}{n}\right)^n = 1 + \sum_{r=1}^{n}\left[\frac{1}{r!}\prod_{k=0}^{r-1}\left(1-\frac{k}{n}\right)\right]$.

   (b) Hence, or otherwise, show that for $n \geq 2$,

   (i) $\displaystyle\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$,

   (ii) $\displaystyle\left(1+\frac{1}{n}\right)^n < 3$.

4.3 Further techniques

4.3.1 Using differentiation and integration

By differentiating and integrating the identity $(1+x)^n = \displaystyle\sum_{r=0}^{n} C_r^n x^r$, we obtain the following useful results:

Theorem 4.6. For any positive integer n,

(a) $\displaystyle n(1+x)^{n-1} = \sum_{r=0}^{n} rC_r^n x^{r-1}$.

(b) $\displaystyle\frac{1}{n+1}\left[(1+x)^{n+1} - 1\right] = \sum_{r=0}^{n}\frac{C_r^n}{r+1}x^{r+1}$.

Proof.

Example 4.5. For any positive integer $n > 1$,

(a) $\displaystyle\sum_{r=0}^{n-1} rC_r^n = n(2^{n-1} - 1)$.

(b) $C_1^n - 2C_2^n + 3C_3^n - 4C_4^n + \cdots + (-1)^{n-1}nC_n^n = 0$.

(c) $C_0^n + \dfrac{1}{2}C_1^n + \dfrac{1}{3}C_2^n + \dfrac{1}{4}C_3^n + \cdots + \dfrac{1}{n+1}C_n^n = \dfrac{2^{n+1}-1}{n+1}$.

Solution.

Example 4.6. Let n be a positive integer. Evaluate

(a) $\displaystyle\sum_{r=0}^{n}(r+1)C_r^n$.

(b) $\displaystyle\sum_{r=0}^{n}(r+1)^2C_r^n$.

(c) $\displaystyle\sum_{r=0}^{n}2r(2r+1)C_r^n$.

Solution.

Example 4.7. Let n be a positive integer. Show that

$$\sum_{r=1}^{n-1} rC_{r+1}^{n} = 1 + (n-2)2^{n-1}.$$

Solution.

Exercise 4.2.

1. Evaluate the following expressions:

   (a) $\displaystyle\sum_{r=0}^{n} (r+2)C_r^n$

   (b) $\displaystyle\sum_{r=0}^{n} \frac{C_r^n}{2r+1}$

   (c) $\displaystyle\sum_{r=1}^{n} (r+1)(r-1)C_{r-1}^n$

   (d) $\displaystyle\sum_{r=0}^{n} \frac{C_r^n}{(r+1)(r+2)}$

   (e) $\displaystyle\sum_{r=0}^{n} \frac{(-1)^r r^2}{r+1} C_r^n$

2. Prove that $\displaystyle\frac{1}{1!(2n)!} + \frac{1}{2!(2n-1)!} + \frac{1}{3!(2n-2)!} + \cdots + \frac{1}{n!(n+1)!} = \frac{2^{2n}-1}{(2n+1)!}.$

4.3.2 Comparing coefficients

Sometimes identities involving binomial coefficients can be proved by expanding an algebraic expression using two different ways and then comparing their coefficients.

Example 4.8. (Vandermonde Theorem)

(a) Let m, n and r be non-negative integers such that $r \leq m+n$. Prove that

$$C_0^m C_r^n + C_1^m C_{r-1}^n + C_2^m C_{r-2}^n + \cdots + C_r^m C_0^n = C_r^{m+n}$$

(b) Hence, show that for any non-negative integer n,

$$(C_0^n)^2 + (C_1^n)^2 + (C_2^n)^2 + \cdots + (C_n^n)^2 = (C_n^{2n})^2$$

Solution.

Example 4.9. By considering the coefficient of x^n in the expansion of $(1-x^2)^n$, show that

$$(C_0^n)^2 - (C_1^n)^2 + (C_2^n)^2 - \cdots + (-1)^n (C_n^n)^2 = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} \frac{n!}{(\frac{n}{2})!(\frac{n}{2})!} & \text{if } n \text{ is even.} \end{cases}$$

Solution.

Exercise 4.3.

1. Using the identity $(1+x)^{2k-p} = (1+x)^{k-p}(1+x)^k$, show that

$$\sum_{r=p}^{k} C^{k-p}_{r-p} C^k_r = C^{2k-p}_k$$

for $0 \le p \le k$.

2. (a) Show that for all positive integers n and m,

$$\frac{(1+x)^{n+m+1} - (1+x)^n}{x} = (1+x)^n + (1+x)^{n+1} + \cdots + (1+x)^{n+m}$$

.

(b) Hence show that $C^n_n + C^{n+1}_n + C^{n+2}_n + \cdots + C^{n+m}_n = C^{n+m+1}_{n+1}$.

(c) Using (b), or otherwise, show that

$$\sum_{r=5}^{m+4} r(r-1)(r-2)(r-3) = 24(C^{m+5}_5 - 1).$$

Hence evaluate $\sum_{r=5}^{m+4} r(r-1)(r-2)(r-3)$ for $k \ge 4$.

3. Suppose n is a positive integer.

(a) Prove that

$$n(1+x)^{2n-1} = \left(\sum_{r=1}^{n} rC^n_r x^{r-1}\right)\left(\sum_{r=0}^{n} C^n_{n-r} x^r\right).$$

(b) Hence by comparing coefficients of a certain power of x, prove that

$$C^n_1 C^n_2 + 2C^n_2 C^n_3 + 3C^n_3 C^n_4 + \cdots + (n-1)C^n_{n-1} C^n_n = \frac{n(2n-1)!}{(n-2)!(n+1)!}.$$