**Proof of Euler's Formula using Infinite Series**

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$i^0 = 1; \quad i = \sqrt{-1}; \quad i^2 = -1; \quad i^3 = -i; \quad i^4 = 1; \quad i^5 = i; \quad etc.$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2} - \frac{ix^3}{6} + \frac{x^4}{24} + \frac{ix^5}{120} - \frac{x^6}{720} - \frac{ix^7}{5040} \ldots$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \ldots$$

$$i\sin(x) = ix - \frac{ix^3}{6} + \frac{ix^5}{120} - \frac{ix^7}{5040} \ldots$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \ldots$$

$$\begin{aligned} \cos(x) + i\sin(x) &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \ldots\right) + \left(ix - \frac{ix^3}{6} + \frac{ix^5}{120} - \frac{ix^7}{5040} \ldots\right) \\ &= 1 + ix - \frac{x^2}{2} - \frac{ix^3}{6} + \frac{x^4}{24} + \frac{ix^5}{120} - \frac{x^6}{720} - \frac{ix^7}{5040} \ldots \\ &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = e^{ix} \end{aligned}$$

*Note : Rearrangement of terms is allowed because the series* $\sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$ *is absolutely convergent.*

$$\therefore e^{ix} = \cos(x) + i\sin(x)$$