MATH 166
THE ADVENTURE BEGINS

This problem is based on a question fielded by Professor Davis Cope in our math department.

On Friday, May 5, 2000 the phone rang in the Math Department. When the secretary answered, a voice told a complicated story about domes and board feet and Hector Airport. ”It sounds like you need calculus,” she said.

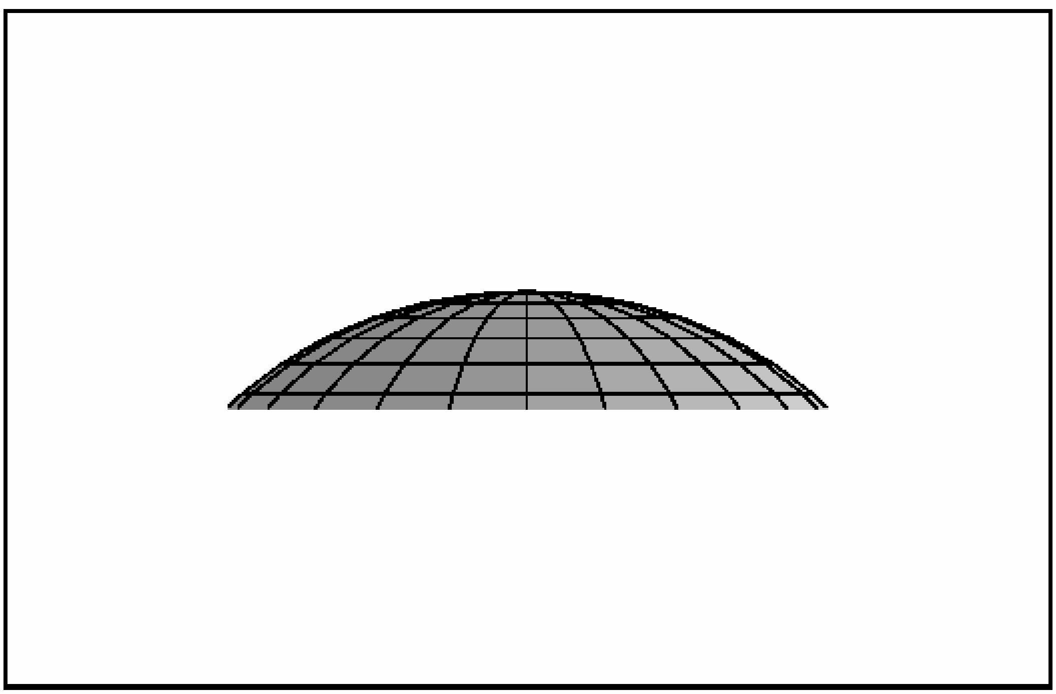
It was Asplin Excavating, preparing an estimate for demolishing the old Valley Aviation Hangar at Hector Airport. The roof of the hangar was like a spherical cap with height 37 feet and with 150 feet as the diameter of the circular base (which is, of course, not necessarily the same as the diameter of the sphere...see picture below).

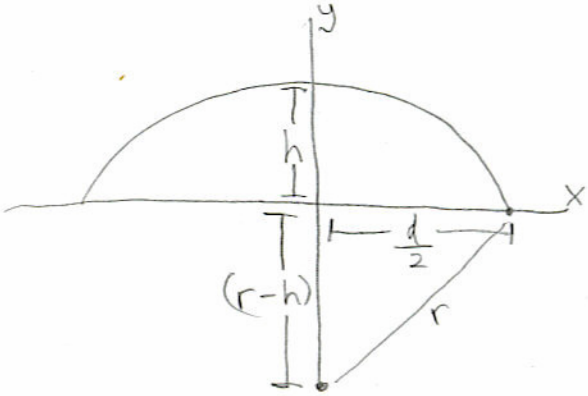
Asplin Excavating needed the area in order to calculate the number of board feet involved (therefore the number of truckloads needed to haul off the junk, which is needed for estimated cost etc.).

The following questions were asked. See if you can answer them.

1. What is the radius of the sphere?
2. What is the area of the roof?
3. Can you find a general formula for the area of a spherical cap when the cap height is h and the spherical radius is R?

Thanks to Professor Davis Cope for telling us about this ”real-life” math story!





$y = \sqrt{r^2 - x^2} - (r-h)$ (circle shifted down)

at $x = \frac{d}{2}, y = 0$

$0 = \sqrt{r^2 - (\frac{d}{2})^2} - (r-h)$; $r^2 - (\frac{d}{2})^2 = (r-h)^2$;

$r^2 - \frac{d^2}{4} = r^2 - 2hr + h^2$; $2hr = \frac{d^2}{4} + h^2$

$$\boxed{r = \frac{h^2 + \frac{d^2}{4}}{2h}}$$

$h = 37,\ d = 150$ $\quad r = \frac{37^2 + \frac{150^2}{4}}{2(37)} = \frac{1369 + 5625}{74} = \frac{6994}{74} = 94.\overline{513}$

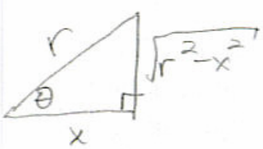
$$\boxed{r = \frac{6994}{74} \text{ ft.}}$$

$S_A = \int_a^b 2\pi x \, ds \qquad ds = \sqrt{1 + (f'(x))^2} \qquad f'(x) = \frac{-2x}{2\sqrt{r^2 - x^2}}$; $(f'(x))^2 = \frac{x^2}{r^2 - x^2}$

spinning the function from 0 to $\frac{d}{2}$ about y-axis will give desired shape.

$$S_A = \int_0^{d/2} 2\pi x \sqrt{1 + \frac{x^2}{r^2 - x^2}} \, dx = 2\pi \int_0^{d/2} x \sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}} \, dx$$

$$= 2\pi \int_0^{d/2} x \frac{r}{\sqrt{r^2 - x^2}} \, dx = 2\pi \int_{\pi/2}^{\cos^{-1}(\frac{d}{2r})} r\cos\theta \frac{r}{r\sin\theta} (-r\sin\theta \, d\theta)$$



$r\sin\theta = \sqrt{r^2 - x^2}$

$r\cos\theta = x$

$dx = -r\sin\theta \, d\theta$

$$= 2\pi \int_{\pi/2}^{\cos^{-1}(\frac{d}{2r})} -r^2\cos\theta \, d\theta = -2\pi r^2 \sin\theta \Big|_{\pi/2}^{\cos^{-1}(\frac{d}{2r})} = -2\pi r \sqrt{r^2 - x^2} \Big|_0^{d/2}$$

$$= -2\pi r \sqrt{r^2 - (\tfrac{d}{2})^2} + 2\pi r^2 = -2\pi r \sqrt{r^2 - 2rh + h^2} + 2\pi r^2$$

from r equation,

$2rh - h^2 = (\frac{d}{2})^2$

$$= -2\pi r \sqrt{(r-h)^2} + 2\pi r^2 = -2\pi r (r-h) + 2\pi r^2$$

$$= -2\pi r^2 + 2\pi r h + 2\pi r^2 = 2\pi r h$$

$$\boxed{S_A = 2\pi r h}$$

$h = 37,\ r = \frac{6994}{74}$ $\quad S_A = 2\pi (\frac{6994}{74})(37) = \boxed{6994\pi \text{ ft.}^2 = S_A}$

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5/5/05