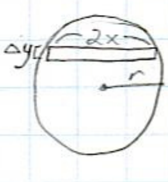
Jordan Dahl 1/29/05

"Volume" of N-D "Sphere"

$$V_2 = \pi r^2 = \int_{-r}^{r} 2(r^2-y^2)^{1/2}\,dy :$$



$x^2+y^2=r^2$

$x=(r^2-y^2)^{1/2}$

$$V_3 = \frac{4}{3}\pi r^3 = \int_{-r}^{r} \pi x^2\,dy = \int_{-r}^{r} \pi (r^2-y^2)\,dy :$$

$$V_4 = \int_{-r}^{r} \frac{4}{3}\pi x^3\,dy = \int_{-r}^{r} \frac{4}{3}\pi (r^2-y^2)^{3/2}\,dy$$

$$= \frac{4}{3}\pi \int_{-r}^{r} \left((r^2-y^2)^{1/2}\right)^3 dy$$

$u = \sin^{-1}\left(\frac{y}{r}\right)$; $\sin u = \frac{y}{r}$

$\cos u\, du = \frac{1}{r}\,dy$; $r\cos u\, du = dy$

$y^2 = r^2\sin^2 u$

$$= \frac{4}{3}\pi r \int_{-\pi/2}^{\pi/2} \left((r^2 - r^2\sin^2 u)^{1/2}\right)^3 \cos u\, du$$

$$= \frac{4}{3}\pi r \int_{-\pi/2}^{\pi/2} \left((r^2(1-\sin^2 u))^{1/2}\right)^3 \cos u\, du = \frac{4}{3}\pi r \int_{-\pi/2}^{\pi/2} (r\cos u)^3 \cos u\, du$$

$$= \frac{4}{3}\pi r^4 \int_{-\pi/2}^{\pi/2} \cos^4 u\, du = \frac{4}{3}\pi r^4 \left[\left(\frac{1}{4}\cos^3\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)\right) - \left(\frac{1}{4}\cos^3\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right)\right) + \frac{3\pi}{8}\right]$$

$$\int_{-\pi/2}^{\pi/2} \cos^4 u\, du = \frac{1}{4}\cos^3 u \sin u\Big]_{-\pi/2}^{\pi/2} + \frac{3}{4}\int_{-\pi/2}^{\pi/2}\cos^2 u\, du = \frac{1}{4}\cos^3 u\sin u\Big]_{-\pi/2}^{\pi/2} + \frac{3}{4}\left(\frac{\pi}{2}\right)$$

$$\int_{-\pi/2}^{\pi/2} \cos^2 u\, du = \frac{1}{2}u + \frac{1}{4}\sin 2u\Big]_{-\pi/2}^{\pi/2} = \left(\frac{\pi}{4} + \frac{1}{4}\sin(\pi)\right) - \left(-\frac{\pi}{4} + \frac{1}{4}\sin(-\pi)\right) = \frac{\pi}{2}$$

$$= \frac{4}{3}\pi r^4 \left(0 + \frac{3\pi}{8}\right) = \frac{1}{2}\pi^2 r^4 = V_4$$

$2 \to 1/2$
$3 \to 1$
$4 \to 3/2$
$(N-1)\left(\frac{1}{2}\right)$

$$V_N = \int_{-r}^{r} k_{(N-1)} (r^2-y^2)^{\left(\frac{N-1}{2}\right)}\,dy = K_N r^N$$

where $k_N = \dfrac{2^{\lceil \frac{N}{2} \rceil}\, \pi^{\lfloor \frac{N}{2} \rfloor}}{N!!}$

where $\lceil x \rceil$ is ceiling function
$\lfloor x \rfloor$ is floor function

$2,3 \to 1$
$4,5 \to 2$
$6,7 \to 3$

$$V_N = \frac{2^{\lceil \frac{N}{2} \rceil}\, \pi^{\lfloor \frac{N}{2} \rfloor}\, r^N}{N!!}$$

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$$V_N = \int_{-r}^{r} k_{(N-1)} \left(r^2 - y^2\right)^{\left(\frac{N-1}{2}\right)} dy \qquad \left(\text{or } 2\int_{0}^{r}\right)$$

This says the coefficient of the volume equation for N dimensions is the coefficient of the volume equation for $N-1$ dimensions multiplied by some new number. As shown for $N=4$, the new multiplier was $3/8$. $\left(\frac{4}{3} \cdot \frac{3}{8} = \frac{1}{2}\right)$ (disregard π)

$$k_N = M_N \, k_{(N-1)}$$

It is interesting how M_N comes about.

for N odd, the exponent $\frac{N-1}{2}$ becomes an integer, making integral much simpler than for N even. Multiplying it out, then, gets into Pascal's triangle: define P_{ij} to be the j^{th} number of the i^{th} row of the triangle.

P_1 — 1
P_2 — 1 1
P_3 — 1 2 1
P_4 — 1 3 3 1
P_5 — 1 4 6 4 1
P_6 — 1 5 10 $\underset{\uparrow \, P_{64}}{10}$ 5 1

After finding the antiderivative, etc., M_N becomes

$$M_N = 2\left(\frac{P_{\left(\frac{N+1}{2}\right)1}}{1} - \frac{P_{\left(\frac{N+1}{2}\right)2}}{3} + \frac{P_{\left(\frac{N+1}{2}\right)3}}{5} - \cdots + \cdots - \cdots\right)$$

ex: $N=9$ $\quad M_9 = 2\left(\frac{1}{1} - \frac{4}{3} + \frac{6}{5} - \frac{4}{7} + \frac{1}{9}\right) = \frac{256}{315} \qquad K_9 = \frac{256}{315} \cdot \underset{k_8}{\frac{1}{24}} = \frac{32}{945}$

for N even, and the ugly integrals because of the integer-and-a-half exponent, M_N comes from $\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$

The limits here would be $a = -\pi/2$ $b = \pi/2$. Notice $\cos(\pm \pi/2) = 0$, making the first term all 0. So all that matters is the second part. Because it becomes $\cos^{n-2} x$, this rule will need to be repeatedly applied until it is $\cos^2 x$, and $\int \cos^2 x \, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$. For the limits in this case, this becomes $\pi/2$. Do not forget the $\frac{n-1}{n}$ back there. Each time it is repeated, the $\frac{n-1}{n}$ is left hanging. Therefore,

$$M_N = \left(\frac{N-1}{N}\left(\frac{N-3}{N-2}\left(\frac{N-5}{N-4}\left(\cdots\left(\underbrace{\frac{N-(N-1)}{N-(N-2)}}_{\frac{1}{2} \text{ (the } \pi/2)}\right)\right)\right)\right)\right) \approx \frac{(N-1)!!}{N!!}$$

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M_2 through M_{10}: Notice patterns (not reduced)

$$\frac{1}{2} \xrightarrow{2} \underset{(3!!)}{\frac{4}{3}} \xrightarrow{2} \frac{3}{8} \xrightarrow{2} \underset{(5!!)}{\frac{16}{15}} \xrightarrow{3} \frac{15}{48} \xrightarrow{2} \underset{(7!!)}{\frac{96}{105}} \xrightarrow{4} \frac{105}{384} \xrightarrow{2} \underset{(9!!)}{\frac{768}{945}} \xrightarrow{5} \frac{945}{3840}$$

to get M_9 $\frac{256}{315}$ alternatively, $\frac{4!\,2^5}{9!!} = \frac{768}{945} = \frac{256}{315}$

so $$M_{\text{odd } N} = \frac{\left(\frac{N-1}{2}\right)!\,2^{\left(\frac{N+1}{2}\right)}}{N!!} \qquad M_{\text{even } N} = \frac{(N-1)!!}{\left(\frac{N}{2}\right)!\,2^{(N/2)}}$$

Recall $k_N = M_N k_{(N-1)}$

$k_{(N-1)} = M_{(N-1)} k_{(N-2)}$; $k_{(N-2)} = M_{(N-2)} k_{(N-3)}$; ...

so $k_N = M_N M_{(N-1)} M_{(N-2)} \cdots M_{(N-(N-2))} k_{(N-(N-1))}$

(there is no k_0)

k_N is the product of all M, M_2 to M_N, and k_1, which is 2 $(2\pi r)$.

$$M_2 M_3 M_4 \ldots = \left[\frac{(2-1)!!}{\left(\frac{2}{2}\right)!\,2^{(2/2)}}\right]\left[\frac{\left(\frac{3-1}{2}\right)!\,2^{\left(\frac{3+1}{2}\right)}}{3!!}\right]\left[\frac{(4-1)!!}{\left(\frac{4}{2}\right)!\,2^{(4/2)}}\right]\cdots$$

notice that much will cancel.
M_2 is $\frac{1}{2}$. Since in the end we must multiply by $k_1 = 2$, we only need $k_N = M_N M_{(N-1)} M_{(N-2)} \cdots M_3$

now $$k_{\text{odd } N} = \frac{2^{\left(\frac{N+1}{2}\right)}}{N!!} \qquad k_{\text{even } N} = \frac{1}{\left(\frac{N}{2}\right)!}$$

$$k_{\text{even } N} = \frac{1}{\left(\frac{N}{2}\right)\left(\frac{N}{2}-1\right)\left(\frac{N}{2}-2\right)\left(\frac{N}{2}-3\right)\cdots} = \frac{1}{\left(\frac{N}{2}\right)\left(\frac{N-2}{2}\right)\left(\frac{N-4}{2}\right)\left(\frac{N-6}{2}\right)\cdots} = \frac{1}{\frac{N!!}{2^{N/2}}}$$
$$= \frac{2^{N/2}}{N!!}$$

to generalize for both odd and even, $$k_N = \frac{2^{\lceil \frac{N}{2} \rceil}}{N!!}$$

to include π, $$k_N = \frac{2^{\lceil \frac{N}{2} \rceil}\,\pi^{\lfloor \frac{N}{2} \rfloor}}{N!!}$$