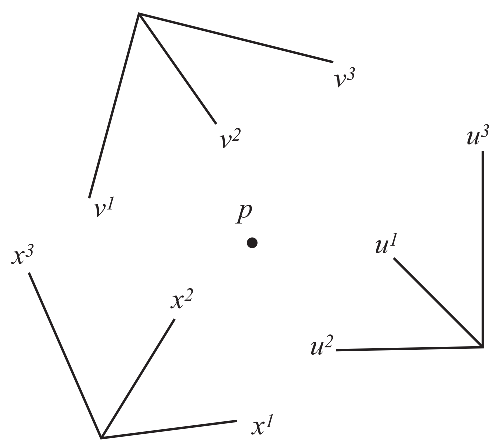

Figure 1: Three different copies of $\mathbb{R}^3$ used to describe the location of a point p.

1 Geometry in $\mathbb{R}^3$

1.1 Coordinates

An element of $\mathbb{R}^3$ is nothing more than an ordered triple (x^1, x^2, x^3), where x^1, x^2, and x^3 are real numbers. As a vector space, $\mathbb{R}^3$ comes with a preferred set of basis elements given by

$$B = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}.$$

On the other hand, $\mathbb{R}^3$ also provides a very convenient means to describe the location of points in a geometrical space. It almost goes without saying that there is an inherent freedom in choosing how to embed the basis elements e_1, e_2, e_3 into the geometrical space. Figure 1 illustrates a point p and three different placements of the basis elements. The location of the point p may be expressed as any of the ordered triples (x^1, x^2, x^3), (u^1, u^2, u^3), or (v^1, v^2, v^3), as long as the choice of how to embed the basis elements in the geometrical space is specified. Obviously, a particular placement of the basis elements in the geometric space amounts to a choice of coordinates.

Once a choice of coordinates has been made, corresponding to an embedding of $\mathbb{R}^3$ into the geometrical space, elements of $\mathbb{R}^3$ are in a one-to-one correspondence with points in the geometrical space. In this way, studying properties of the geometrical space amounts to studying properties of $\mathbb{R}^3$. In fact, in what follows, $\mathbb{R}^3$ will be identified with the geometrical space, although it should be remembered that $\mathbb{R}^3$ is really just a description in terms of a choice of coordinates and any other choice would have sufficed just as well.

When it is desired to make the distinction between $\mathbb{R}^3$ and the geometrical space clear, the geometrical space will be denoted by M with points in M denoted by p. A choice of coordinates can then be denoted by a coordinate map $\phi : M \to \mathbb{R}^3$ defined by

$$\phi(p) = (x^1(p), x^2(p), x^3(p)), \tag{1}$$

where $x^i : M \to \mathbb{R}$ are the coordinate functions. Different coordinate maps $M \to \mathbb{R}^3$ then correspond to different choices of coordinates.

1.2 Curves and Tangent Vectors

A curve in a geometrical space M is simply a smooth map $\gamma : \mathbb{R} \to M$. The rate of change of γ with respect to t at t_0 is given by

$$\left.\frac{d\gamma}{dt}\right|_{t_0} := \lim_{\Delta t \to 0} \frac{\gamma(t_0 + \Delta t) - \gamma(t_0)}{\Delta t}. \tag{2}$$

This rate of change is a vector at $\gamma(t_0) \in M$ that is tangent to the curve and may be thought of as a velocity vector.

As stressed in the previous section, however, once a choice of coordinate maps ϕ has been made, it is constructive to consider curves in $\mathbb{R}^3$. Having chosen a coordinate map ϕ, a curve γ in M implies a curve $\tilde{\gamma} := \phi \circ \gamma$ in $\mathbb{R}^3$. The rate of change of $\tilde{\gamma}$ with respect to t at t_o is similarly given by

$$\begin{aligned}
\left.\frac{d\tilde{\gamma}}{dt}\right|_{t_0} &= \left.\frac{d(\phi \circ \gamma)}{dt}\right|_{t_0} \\
&= \lim_{\Delta t \to 0} \frac{\phi \circ \gamma(t_0 + \Delta t) - \phi \circ \gamma(t_0)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left(\frac{\tilde{x}^1(t_0 + \Delta t) - x^1(t_0)}{\Delta t}, \frac{\tilde{x}^2(t_0 + \Delta t) - x^2(t_0)}{\Delta t}, \frac{\tilde{x}^3(t_0 + \Delta t) - x^3(t_0)}{\Delta t} \right) \\
&= \left.\left(\frac{d\tilde{x}^1}{dt}, \frac{d\tilde{x}^2}{dt}, \frac{d\tilde{x}^3}{dt} \right)\right|_{t_0} \\
&= \left.\frac{d\tilde{x}^i}{dt}\right|_{t_0} e_i,
\end{aligned} \tag{3}$$

where $\tilde{x}^i = x^i \circ \gamma$ and e_i is the natural basis element of $\mathbb{R}^3$. This is clearly a vector that is also tangent to the curve $\tilde{\gamma}$ in $\mathbb{R}^3$. Consequently, the coordinate map not only provides a mapping of points in M to points in $\mathbb{R}^3$, it also induces a map from tangent vectors in M to tangent vectors in $\mathbb{R}^3$. The induced map of tangent vectors is denoted by

$$\phi_* \left.\frac{d\gamma}{dt}\right|_{t_0} := \left.\frac{d(\phi \circ \gamma)}{dt}\right|_{t_0} \tag{4}$$

and is called the push forward of $\left.\frac{d\gamma}{dt}\right|_{t_0}$ from M to $\mathbb{R}^3$.

Recall that $\mathbb{R}^3$ may be thought of as a geometrical space in its own right, and hence ϕ is a map from one geometrical space to another. The general concept of pushing forward tangent vectors extends in a straight-forward manner to any smooth map from one geometrical space to another.

Definition 1 *If M and N are two geometrical spaces, $F : M \to N$ is a smooth map, and $\gamma : \mathbb{R} \to M$ is a curve in M, then $\left.\frac{d\gamma}{dt}\right|_{t_0}$ is a tangent vector at the point $\gamma(t_0) \in M$ and the push forward of this tangent vector to a tangent vector at the point $F \circ \gamma(t_0) \in N$ is given by*

$$F_* \left.\frac{d\gamma}{dt}\right|_{t_0} := \left.\frac{d(F \circ \gamma)}{dt}\right|_{t_0}. \tag{5}$$

This general view of the push forward of tangent vectors provides another interpretation of $\left.\frac{d\gamma}{dt}\right|_{t_0}$. Consider

$$\begin{aligned}
\left.\frac{d\gamma}{dt}\right|_{t_0} &= \left.\frac{d(\gamma \circ \mathrm{id})}{dt}\right|_{t_0} \\
&= \gamma_* \left.\frac{d(\mathrm{id})}{dt}\right|_{t_0} \\
&= \gamma_* \lim_{\Delta \to 0} \frac{\mathrm{id}(t_0 + \Delta t) - \mathrm{id}(t_0)}{\Delta t} \\
&= \gamma_* \left(1|_{t_0}\right) . \qquad (6)
\end{aligned}$$

In other words, the tangent vector $\left.\frac{d\gamma}{dt}\right|_{t_0}$ is simply the push forward of the tangent vector 1 at t_0 from $\mathbb{R}$ to M via the curve γ. Also, the curve $\tilde{\gamma} = \phi \circ \gamma : \mathbb{R} \to \mathbb{R}^3$ may be written as

$$\left.\frac{d\tilde{\gamma}}{dt}\right|_{t_0} = \tilde{\gamma}_* \left(1|_{t_0}\right) = (\phi \circ \gamma)_* \left(1|_{t_0}\right) . \qquad (7)$$

However,

$$\left.\frac{d\tilde{\gamma}}{dt}\right|_{t_0} = \phi_* \left.\frac{d\gamma}{dt}\right|_{t_0} = \phi_* \circ \gamma_* \left(1|_{t_0}\right) , \qquad (8)$$

which shows that $(\phi \circ \gamma)_* = \phi_* \circ \gamma_*$.

As another special case of the push forward of tangent vectors, consider a smooth map $f : M \to \mathbb{R}$, i.e. a scalar function on M, and a curve $\gamma : \mathbb{R} \to M$, then the push forward of $\left.\frac{d\gamma}{dt}\right|_{t_0}$ from M to $\mathbb{R}$ via f is given by

$$f_* \left.\frac{d\gamma}{dt}\right|_{t_0} = \left.\frac{d(f \circ \gamma)}{dt}\right|_{t_0} . \qquad (9)$$

On the other hand, the directional derivative D_γ of f along the curve γ at the point $\gamma(t_0) \in M$ is defined by

$$D_\gamma f \Big|_{t_0} := \left.\frac{d(f \circ \gamma)}{dt}\right|_{t_0} . \qquad (10)$$

Hence, the directional derivative of a function along a curve corresponds simply to pushing forward the tangent vector of the curve to $\mathbb{R}$ via that function.