MATH 5392 – ALGEBRAIC GEOMETRY

JEFF GARRETT

5.11

1. If $\mathbb{A}^n_{(0)}$ and $\mathbb{A}^m_{(0)}$ are standard affine pieces of $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively, verify that $\mathbb{A}^n_{(0)} \times \mathbb{A}^m_{(0)}$ is mapped isomorphically to an affine piece of the variety $S_{n,m} \subset \mathbb{P}^N$, say $S_{(0)} \subset \mathbb{A}^N$, and that the N coordinates of $\mathbb{A}^N$ restrict to $X_1, \dots, X_n, Y_1, \dots, Y_m$ and nm terms X_iY_j.
2. If $V \subset \mathbb{P}^n$ and $W \subset \mathbb{P}^m$, prove that the product $V \times W$ is a projective subvariety of $\mathbb{P}^n \times \mathbb{P}^m = S_{n,m} \subset \mathbb{P}^N$.

---

1. Let $\mathbb{A}^n_{(0)} = \{[a_0 : a_1 : \cdots : a_n] \in \mathbb{P}^n \,|\, a_0 \neq 0\}$ and similarly for $\mathbb{A}^m_{(0)}$. If $\phi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ is the Segre embedding and $S_{n,m}$ is the image of ϕ, it is clear that $\mathbb{A}^n_{(0)} \times \mathbb{A}^m_{(0)}$ is mapped isomorphically under ϕ to $\mathbb{A}^{nm+n+m}_{(0)} \cap S_{n,m}$ by the definition of ϕ. This is an affine piece $S_{(0)}$ of $S_{n,m}$. Once again appealing to the definition of ϕ we know that the coordinates of a point in $S_{(0)} \subset \mathbb{A}^{nm+n+m}$ are of the form $(a_ib_j)/(a_0b_0) = (a_i/a_0)(b_j/b_0)$ which consist of the required form if $X_i = a_i/a_0$ and $Y_j = b_j/b_0$.
2. If F_i are homogeneous polynomials of degrees d_i which define $V \subset \mathbb{P}^n$ then $F_iY_j^{d_i}$ are homogeneous polynomials in terms of $X_kY_j = Z_{k,j}$ where $Z_{k,j}$ ($k = 0, \dots, n$ and $j = 0, \dots, m$) are the coordinates in $\mathbb{P}^{nm+n+m}$. We do a similar construction for the polynomials which cut out W and we add the polynomials $Z_{i,j}Z_{k,l} - Z_{i,k}Z_{j,l}$ (which define $S_{n,m}$) to get $V \times W \subset S_{n,m}$.

Date: September 30, 1999.