MATH 5392 – ALGEBRAIC GEOMETRY

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5.1 Prove that a regular function on $\mathbb{P}^1$ is a constant. Deduce that there are no non-constant morphisms $\mathbb{P}^1 \to \mathbb{A}^m$ for any m.

If one restricts attention to the two affine pieces $\mathbb{A}^1_{(0)}$ and $\mathbb{A}^1_{(1)}$ then a regular function f on $\mathbb{P}^1$ is regular on these affine pieces. If we assume (as we do here) that our field is algebraically closed this means that f must be a polynomial on each of these affine pieces. That is for all $[1 : y]$ and $[x : 1]$ in the respective affine pieces $f(y) = p(y)$ and $f(x) = q(x)$ for some polynomials p and q. But on the intersection of the affine pieces we have $x = 1/y$ so that $p(y) = q(1/y)$ for all $[1 : y]$ such that $y \neq 0$. Multiplying by an appropriate power of y to clear the denominators on the right, we have a polynomial which is 0 for infinitely many values in k. This is a contradiction unless the polynomial is 0. But the constant term is the only one for which the coefficients of p and q may cancel. This means both p and q must be a constant, and moreover the same constant.

That there are no non-constant morphisms $\mathbb{P}^1 \to \mathbb{A}^m$ follows from the definition of a morphism as a function $p \mapsto (f_1(p), \ldots, f_m(p))$ where $f_1, \ldots, f_m$ are regular functions on $\mathbb{P}^1$.

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