MATH 5392 – ALGEBRAIC GEOMETRY

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3.6 Let $\Bbbk$ be a field and $J = (X^2 + Y^2 - 1, Y - 1) \subset \Bbbk[X, Y]$. Find $f \in \mathrm{I}(\mathrm{V}(J)) \setminus J$.

First $J = (X^2, Y - 1)$ since $X^2 + Y^2 - 1 = X^2 + (Y + 1)(Y - 1)$. Thus it is apparent $\mathrm{V}(J) = \{(0, 1)\}$. Hence, $\mathrm{I}(\mathrm{V}(J)) = (X, Y - 1)$. So $f(X, Y) = X \in \mathrm{I}(\mathrm{V}(J)) \setminus J$. To show that $X \notin J$, consider linear combinations of X^2 and $Y - 1$. Expanding any such polynomial in terms of powers of X, the coefficient of X is a multiple of $Y - 1$. Since $Y - 1$ is not a unit, $X \notin J$.

3.13a Let $\Bbbk$ be an infinite field and $f \in \Bbbk[X_1, \ldots, X_n]$. Suppose that f is nonconstant. Prove that $\mathrm{V}(f) \neq \Bbbk^n$.

If $n = 1$, then $f \in \Bbbk[X_1]$ is a polynomial over $\Bbbk$, so that it has at most n roots. Choose an element p of the infinite field $\Bbbk$ which is not a root. Then $p \notin V(f)$.

Now assume that it has been proven for less than n indeterminates. If f doesn't involve X_n we may use the inductive hypothesis to achieve our conclusion. Otherwise, we may write $f = \sum a_i(X_1, \ldots, X_{n-1})X_n^i$. Let X_n^m ($m \geq 1$) be the highest power of X_n appearing in this expression. Then by the inductive hypothesis, there exist $p_1, \ldots, p_{n-1} \in \Bbbk$ such that $a_m(p_1, \ldots, p_{n-1}) \neq 0$. Then $f(p_1, \ldots, p_{n-1}, X_n)$ is a polynomial in X_n of degree m. Therefore it has at most m roots, and we may choose $p_n \in \Bbbk$ such that p_n is not one of the roots, since $\Bbbk$ is infinite. Then $f(p_1, \ldots, p_n) \neq 0$ and $(p_1, \ldots, p_n) \notin V(f)$. Thus $V(f) \neq \Bbbk^n$.

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