MATH 5392 – ALGEBRAIC GEOMETRY

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4.6	1. Show that the domain of $g \circ f$ may be strictly larger than $\operatorname{dom} f \cap f^{-1}(\operatorname{dom} g)$. 2. Most courses on calculus of several variables contain examples such as the function $f(x, y) = \frac{xy}{x^2+y^2}$. Explain why f is C^∞ when restricted to any smooth curve through $(0, 0)$, but is not even continuous as a function of two variables.

1. Let $C = \mathrm{V}(Y^2 - X^3) \subset \mathbb{A}^2_k$. Let $f : \mathbb{A}^1_k \to C$ be given by $T \mapsto (T^2, T^3)$. Let $g : C \to \mathbb{A}^1_k$ be given by $(X, Y) \mapsto Y/X$. Then $\operatorname{dom} f = \mathbb{A}^1_k$ and $\operatorname{dom} g = C \setminus \{(0, 0)\}$. Also $g \circ f : T \mapsto T^3/T^2 = T$ (as a member of $k(T)$). Thus $\operatorname{dom} g \circ f = \mathbb{A}^1_k$ and $\operatorname{dom} f \cap f^{-1}(\operatorname{dom} g) = \mathbb{A}^1_k \setminus \{(0, 0)\}$, two distinct sets.
2. f is not continuous because the limit of $f(x, y)$ for $x = 0$ and $y \to 0$ is 0, and the limit as $x = y$ and $y \to 0$ is $\frac{1}{2}$. Parameterize any smooth curve through the origin by arc length as $x = tm(t)$ and $y = tn(t)$. Suppose that the origin corresponds to $t = 0$. One of $m(0)$ or $n(0)$ is nonzero, assume it is $m(0)$. Then
$$f(x, y) = \frac{n/m}{1 + (n/m)^2}$$
Expanding it as a taylor series about $t = 0$, one sees that it is C^∞.

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