MATH 5392 – ALGEBRAIC GEOMETRY

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4.11 Define and study the notion of product of two algebraic sets. More precisely,

1. if $V \subset \mathbb{k}^n$ and $W \subset \mathbb{k}^m$ are algebraic sets, prove that $V \times W \subset \mathbb{k}^{n+m}$ is also;
2. give examples to show that the Zariski topology on $V \times W$ is not the product topology of those on V and W;
3. prove that V, W irreducible implies $V \times W$ is irreducible;
4. prove that if $V \cong V'$ and $W \cong W'$ then $V \times W \cong V' \times W'$.

For $V \subset \mathbb{k}^n$, $W \subset \mathbb{k}^m$ algebraic subsets let us define $V \times W = \{(v, w) \mid v \in V, w \in W\} \subset \mathbb{k}^{n+m}$.

1. Let $J = \mathrm{I}(V) \subset \mathbb{k}[X_1, \cdots, X_n]$ and $K = \mathrm{I}(W) \subset \mathbb{k}[Y_1, \cdots, Y_m]$. Then considering J and K as subsets of $\mathbb{k}[X_1, \cdots, X_n, Y_1, \cdots, Y_m]$, we may take the ideals generated by J, K in this ring, and denote them J' and K'. Looking at generators of J and K it is apparent that $\mathrm{V}(J') = V \times \mathbb{k}^m$ and $\mathrm{V}(K') = \mathbb{k}^n \times W$. Then $\mathrm{V}(J' + K') = \mathrm{V}(J') \cap \mathrm{V}(K') = V \times W$. Thus $V \times W$ is an algebraic set.
2. Let $\mathrm{V}(X - Y) = \Delta \subset \mathbb{k}^2$. Then Δ is closed in the Zariski topology. If Δ were closed in the product, then it is the complement of a union of sets of the form $U_i \times V_i$. In particular a set of this form must lie in the complement of Δ. Assuming $\mathbb{k}$ is algebraically closed, then U_i and V_i have finite complement in $\mathbb{k}$. Since $\mathbb{k}$ is infinite, U_i and V_i must have a point m in common. Then $(m, m) \in U_i \times V_i$ but $(m, m) \in \Delta$. This is a contradiction, hence Δ is not closed in the product topology.
3. If $V \times W = Z_1 \cup Z_2$, then consider $\{v\} \times W$ for arbitrary $v \in V$. This is closed and $\{v\} \times W_i = \{v\} \times W \cap Z_i$ is as well. Suppose $\{v\} \times W_1 = \mathrm{I}(J)$ for $J = (f_1, \ldots, f_s)$. Then $W_1 = \mathrm{V}(f_1(v, Y), \ldots, f_s(v, Y))$ is closed. Similarly W_2 is closed. Since $W = W_1 \cup W_2$, either $W = W_1$ or $W = W_2$. Then consider $V_i = \{v \mid \{v\} \times W \subset Z_i\}$. $V = V_1 \cup V_2$. Since $v \in V_i$ if and only if $f(v, w) = 0$ for all $f \in \mathrm{I}(Z_i)$ and $w \in W$, it is closed, being the zero locus of polynomials. So $V = V_1$ or V_2. That is, either $V \times W \subset Z_1$ or $V \times W \subset Z_2$.
4. Let $f : V \to V'$ be a polynomial map with inverse f'. Similarly let $g : W \to W'$ be an isomorphism of varieties with inverse g'. Define $f \times g : V \times W \to V' \times W'$ as $(f \times g)(v, w) = (f(v), g(w))$. Then $f \times g$ is a polynomial map and in addition, $(f \times g)(f' \times g') = 1_{V' \times W'}$ and $(f' \times g')(f \times g) = 1_{V \times W}$. Thus $V \times W \cong V' \times W'$.

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