MATH 5392 – ALGEBRAIC GEOMETRY

JEFF GARRETT

3.4 Let A be an integral domain with field of fractions K. Let $0 \notin S \subset A$ and let $B = A[S^{-1}]$ where $S^{-1} = \{s^{-1} \mid s \in S\} \subset K$. Show that ideals of B are uniquely determined by their intersection with A. Further prove that if A is Noetherian, then so is B.

Without loss of generality we may assume that $1 \in S$ and that S is multiplicative ($x, y \in S \Rightarrow xy \in S$). If S doesn't satisfy these requirements we can consider a minimal set $\bar{S}$ containing S which does. We may take $\bar{S}$ to be the set consisting of finite products of elements of S together with 1. Then it is clear $B = A[S^{-1}] = A[\bar{S}^{-1}]$.

With this assumption, every element of B may be written in the form $\frac{a}{s}$ with $a \in A$ and $s \in S$. Further for an ideal I of B we denote $I \cap A$ by I^{c}. For an ideal J of A we denote JB, the ideal generated in B by J, by J^{e}.

Lemma. *If I is an ideal of B, then $I^{\mathrm{ce}} = I$.*

Proof. First, $I^{\mathrm{ce}} = (I \cap A)B \subset IB = I$. Also, if $\frac{a}{s} \in I$, then $\frac{a}{s}s = a \in I$ and so $a \in I \cap A = I^{\mathrm{c}}$. Thus $a\frac{1}{s} = \frac{a}{s} \in I^{\mathrm{c}}B = I^{\mathrm{ce}}$. This shows the equality. □

From this it follows that ideals of B are determined by their intersection with A. For if I, J are ideals, and $I^{\mathrm{c}} = I \cap A = J \cap A = J^{\mathrm{c}}$ then $I = I^{\mathrm{ce}} = J^{\mathrm{ce}} = J$.

Now, if A is Noetherian, it satisfies the ascending chain condition. Further if $J_0 \subset J_1 \subset J_2 \subset J_3 \cdots$ is an ascending chain of ideals of B, then $J_0^{\mathrm{c}} \subset J_1^{\mathrm{c}} \subset J_2^{\mathrm{c}} \subset J_3^{\mathrm{c}} \cdots$ is an ascending chain of ideals of A and thus there exists an N such that $J_N^{\mathrm{c}} = J_{N+1}^{\mathrm{c}} = J_{N+2}^{\mathrm{c}} = \cdots$. Since ideals of B are determined by their intersection with A, it follows that $J_N = J_{N+1} = J_{N+2} = \cdots$. Thus B satisfies the ascending chain condition, and is Noetherian.

Date: August 31, 1999.