MATH 5392 – ALGEBRAIC GEOMETRY

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1.11

1. Show by example that the set difference of two affine varieties $V \setminus W$ need not be an affine variety.
2. Show that for any ideals I, J in $k[X_1, \dots, X_n]$, $\mathrm{V}(I : J)$ contains $\mathrm{V}(I) \setminus \mathrm{V}(J)$, but that we may not have equality.
3. If I is a radical ideal, show that any algebraic variety containing $\mathrm{V}(I) \setminus \mathrm{V}(J)$ must contain $\mathrm{V}(I : J)$.
4. Show that if I is a radical ideal and J is any ideal, then $I : J$ is also a radical ideal. Deduce that $\mathrm{I}(V) : \mathrm{I}(W)$ is the radical ideal corresponding to $\overline{V \setminus W}$ in the ideal-variety correspondence.

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1. The hint given suggests that $V = k$ and $W = \{0\}$. Then $V \setminus W = k \setminus \{0\}$ is not an algebraic set in $\mathbb{A}^1$ and hence not an affine variety in the sense of this exercise, because all algebraic sets in $\mathbb{A}^1$ are either finite or all of $\mathbb{A}^1$ and k can be assumed infinite. However $V \setminus W$ is an affine variety in our earlier, broader, sense with a coordinate ring of $k[x, x^{-1}] = k[x, y]/(xy - 1)$. It can be realized as the algebraic set $\mathrm{V}(xy - 1) \subset \mathbb{A}^2$.
2. If $x \notin \mathrm{V}(I : J)$ and $x \notin \mathrm{V}(J)$ then there exist $f \in (I : J)$ and $g \in J$ with $g(x) \neq 0$ and $f(x) \neq 0$. By definition however $gf \in I$. So $(gf)(x) \neq 0$ and $x \notin \mathrm{V}(I)$. This shows $\mathrm{V}(I) \setminus \mathrm{V}(J) \subset \mathrm{V}(I : J)$.

   An example where equality is not achieved is the same example as in the above, $k \setminus \{0\}$. For instance, let $I = (0)$ and $J = (x)$. Then $I : J = (0)$ and $\mathrm{V}(I) \setminus \mathrm{V}(J) = k \setminus \{0\}$ but $\mathrm{V}(I : J) = k$.
3. This is equivalent to the statement $M = \mathrm{I}(\mathrm{V}(I) \setminus \mathrm{V}(J)) = I : J$ with the help of the next problem. But $I : J \subset M$ because for any $h \in I : J$ and $x \in \mathrm{V}(I)$ but not in $\mathrm{V}(J)$ we know that there is an $f \in J$ such that $f(x) \neq 0$. But $hf \in I$ so that $h(x)f(x) = 0$, and hence $h(x) = 0$.

   To get the converse, suppose $f \notin I : J$ and $f \in M$. Then there is an $h \in J$ such that $fh \notin I$. Since $h(x) = 0$ for all $x \in \mathrm{V}(J)$ and $f(x) = 0$ for all $x \in \mathrm{V}(I) \setminus \mathrm{V}(J)$ we know that $(hf)(x) = 0$ for all $x \in \mathrm{V}(I)$ which implies since I is radical (by the Nullstellensatz) that $hf \in I$, a contradiction.

   To see the equivalence more clearly, suppose $\mathrm{V}(W)$ contains $\mathrm{V}(I) \setminus \mathrm{V}(J)$. Without loss of generality we may assume that W is radical. By applying I we get that $W \subset M = I : J$. Thus $\mathrm{V}(I : J) \subset \mathrm{V}(W)$. More succinctly we could have recalled that if $X \subset \mathbb{A}^n$, then $\mathrm{V}(\mathrm{I}(X)) = \overline{X}$. This implies that $\overline{\mathrm{V}(I) \setminus \mathrm{V}(J)} = \mathrm{V}(I : J)$.
4. Suppose I is a radical ideal and $g^n \in (I : J)$. Then for all $h \in J$, we know $g^n h \in I$. Specifically for all $h \in J$, we have $h^n \in J$ so that $(gh)^n = g^n h^n \in I$. But I is radical, so we have for all $h \in J$ that $gh \in I$. This shows that $(I : J)$ is radical.

Date: November 9, 1999.