MATH 5392 – ALGEBRAIC GEOMETRY

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4.2

1. Find all points in $V = \mathrm{V}(x^2-2x+y^2, x^2-4x+4y^4) \subset \mathbb{C}^2$ and compute the multiplicity of each as above.
2. Verify that the sum of the multiplicities is equal to $\dim \mathbb{C}[x,y]/(x^2-2x+y^2, x^2-4x+4y^4)$.
3. What is the geometric explanation for the solution of multiplicity greater than 1 in this example?

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1. A Gröbner basis for $(x^2-2x+y^2, x^2-4x+4y^4)$ is given by the polynomials $-2x+4y^4-y^2$ and $y^2(4y^6-2y^4-\frac{15}{4}y^2+2)$ where the sextic is irreducible over $\mathbb{Q}$. Thus there are three values of y^2 (for $y \neq 0$) and for each one there is a unique x which will make these polynomials zero simultaneously. This gives six points (because $\mathbb{Q}$ is a perfect field). The origin is also a point, yielding seven total. To be more precise, the points in V are $(0,0)$ and $(2y^4-\frac{1}{2}y^2, y)$ where y is any root of the above sextic.
2. From the above given Gröbner basis, it is clear that $1, y, \ldots, y^7$ are a basis of $A = \mathbb{C}[x,y]/(x^2-2x+y^2, x^2-4x+4y^4)$ so that $\dim A = 8$. Further, multiplication by x acts as a linear map on A. Through a simple calculation, one finds that none of $x, xy, \ldots, xy^7$ contain a 1 or y term when expanded in terms of the basis. Thus the first two columns of the matrix representing this map are 0, and so the characteristic polynomial $\chi_x(t)$ is divisible by t^2. Thus the multiplicity of the origin is 2 and it follows that the other points have multiplicity 1, which adds to 8 total.
3. The curves are mutually tangent at the origin (because of the symmetry about the x-axis, they have vertical tangents) and thus the origin is a point of contact of order 2.

4.11 Let I be a zero-dimensional ideal with $\mathrm{V}(I) = \{p_1, \ldots, p_m\}$. Let $A = k[x_1, \ldots, x_n]/I$ and $A_i = k[x_1, \ldots, x_n]_{p_i}/Ik[x_1, \ldots, x_n]_{p_i}$. Furthermore, let $\phi_i : k[x_1, \ldots, x_n] \to A_i$ be the composition of the natural inclusion $k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]_{p_i}$ followed by the natural epimorphism onto its factor ring. Let us denote the action of ϕ_i by $f \mapsto [f]_i$. Let $Q_i = \ker \phi_i$. And let $M_i = \mathrm{I}(\{p_i\})$.

1. Show that $I \subset Q_i$ and $Q_i = \{f \mid uf \in I \text{ for some } u \in k[x_1, \ldots, x_n] \setminus M_i\}$.
2. If $g_1, \ldots, g_m$ are such that $g_i(p_i) = 1$ and $g_i(p_j) = 0$ for $i \neq j$, show that for $j \neq i$ some power of g_j lies in Q_i.
3. Show that $\mathrm{V}(Q_i) = \{p_i\}$ and conclude that $\sqrt{Q_i} = M_i$.
4. Show that Q_i is a primary ideal.
5. Prove that $I = Q_1 \cap Q_2 \cap \cdots \cap Q_m$.
6. Show that $k[x_1, \ldots, x_n]/Q_i \cong A_i$.

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1. That $I \subset Q_i$ follows from the definition of ϕ_i. Furthermore $f \in Q_i = \ker \phi_i$ if and only if $f = \frac{g}{h}$ in $k[x_1, \ldots, x_n]_{p_i}$ for some $g \in I$, and $h \in k[x_1, \ldots, x_n] \setminus M_i$. This can happen if and only if $fh = g$ in $k[x_1, \ldots, x_n]$. Thus the statement is proved.
2. By the above, characterization, it follows that $\mathrm{V}(Q_i) = \{p_i\}$ because any $f \in Q_i$ must be 0 at p_i and furthermore, if one chooses a function f such that $f(p_j) \neq 0$ (for $j \neq i$) and $f(p_i) = 0$, say the sum of the g_j for $j \neq i$, and a function u such that $u(p_j) = 0$ for $j \neq i$ but $u(p_i) \neq 0$ (so that $u \notin M_i$), say g_i, it follows that $(uf)(p_j) = 0$ for any p_j so that $(uf)^n \in I$ for some n (by the Nullstellensatz) and by the above characterization of Q_i, $f^n \in Q_i$. By this construction it follows that $\mathrm{V}(Q_i) = \{p_i\}$. The

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important fact for this exercise is that $p_j \notin \mathrm{V}(Q_i) \subset \mathrm{V}(I)$ however. This means that $g_j \in \mathrm{I}(\mathrm{V}(Q_i))$ so that $g_j^n \in Q_i$ for some n.

3. I showed that $\mathrm{V}(Q_i) = \{p_i\}$ in the above. Since $\mathrm{V}(Q_i) = \{p_i\}$, by the Nullstellensatz, $\sqrt{Q_i} = M_i$.
4. Suppose that $fg \in Q_i$. Then $\phi_i(fg) = \phi_i(f)\phi_i(g) = 0$. If $g^n \notin Q_i$ for any n then by the above $g \notin M_i$ so that $\phi_i(g)$ is a unit, and $\phi_i(f) = 0$ so that $f \in Q_i$.
5. We have already proved that $\ker \phi = I$ where $\phi : k[x_1, \ldots, x_n] \to A_1 \times \cdots \times A_m$ is defined by $\phi : f \mapsto (\phi_1(f), \ldots, \phi_m(f))$. The kernel is precisely $Q_1 \cap \cdots \cap Q_m$, whence the proposition follows.
6. All that is necessary is to show that ϕ_i is surjective. But we have already proved that ϕ (as in the above) is surjective. It follows that ϕ_i is surjective