MATH 5392 – ALGEBRAIC GEOMETRY

JEFF GARRETT

6.3 Determine all the singular points of the following curves in $\mathbb{A}^2$.

1. $y^2 = x^3 - x$
2. $y^2 = x^3 - 6x^2 + 9x$
3. $x^2y^2 + x^2 + y^2 + 2xy(x + y + 1) = 0$
4. $x^2 = x^4 + y^4$
5. $xy = x^6 + y^6$
6. $x^3 = y^2 + x^4 + y^4$
7. $x^2y + xy^2 = x^4 + y^4$

---

1. The polynomial $F = y^2 - x(x-1)(x+1)$ is irreducible in the ring $k[x, y] = k[x][y]$ by Eisenstein's criterion with the prime $x \in k[x]$. Moreover,
$$\frac{\partial F}{\partial y} = 2y$$
$$\frac{\partial F}{\partial x} = -3x^2 + 1$$
Assuming $\operatorname{char} k \neq 2$, these polynomials cannot be zero simultaneously. Thus $V_{sing} = \varnothing$.
2. By the same argument, $F = y^2 - x(x-3)^2$ is irreducible in $k[x, y]$. Also
$$\frac{\partial F}{\partial x} = -(x-3)^2 - x(x-3) = (x-3)(-2x+3)$$
$$\frac{\partial F}{\partial y} = 2y$$
Assuming $\operatorname{char} k \neq 2$ we have $V_{sing} = \{(3, 0)\}$.
3. The polynomial $x^2y^2 + x^2 + y^2 + 2xy(x + y + 1)$ is the square of $F = x + y + xy$ which is irreducible by the same argument as above. Further the partial derivatives are $1 + x$ and $1 + y$. Since $x = -1$ and $y = -1$ is not a root of F we have that $V_{sing} = \varnothing$.
4. The polynomial $F = x^2(x-1)(x+1) + y^4$ is irreducible by Eisenstein with the prime $x - 1$ (assuming $\operatorname{char} k \neq 2$). The partial derivatives are $2x(1 - 2x^2)$ and $4y^3$. The only common root of all of these polynomials is $(0, 0)$ so $V_{sing} = \{(0, 0)\}$.
5. The polynomial $F = x^6 + y^6 - xy$ is irreducible and the ideal generated by F, $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ has a Gröbner basis (with respect to a Lex ordering with $x > y$) consisting of $-x + 6y^5$ and $\frac{-1}{46656}y$. Therefore the only singular point is $(0, 0)$.
6. A Gröbner basis (with Lex for $x > y$) for the relevant ideal consists of $\frac{3}{16}x^2 - \frac{1}{2}y^2$ and $\frac{91}{48}y$. Thus the only singular point is the origin.
7. Again an element of a Gröbner basis of the relevant ideal contains $\frac{-1}{72}y^3$ which implies that the origin is the only singular point.

Date: October 12, 1999.