The Recovery of Rockets

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1. Introduction

The primary objective of any recovery sequence, of course, is the reduction of the rockets velocity to some value for which adequate impact survival may be obtained. The parachute is ejected close to apogee due to the relative low velocity, to avoid damage to the recovery system and the rocket itself. Typically a cross-type parachute consisting of five squares is used because it is easy to fabricate. The length of all the rigging lines are equal to two times the side in one square.

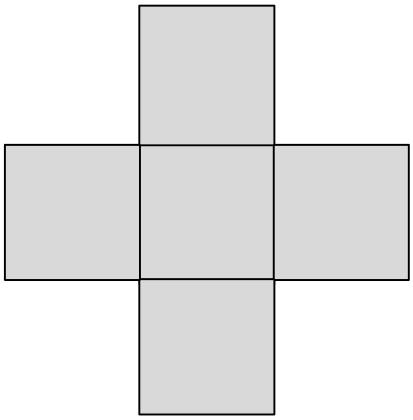

Fig. 1 Cross-type parachute

A timer is use as the sole method to initiate the recovery. The timer is set to a predetermined time based on the trajectory calculations. At take-off the timer is initiated by use of an acceleration switch. The timer for the recovery system is self-contained with its own power supply. This design assures a reliable and safety launch procedure. When the rocket is mounted on the launcher, the only requirement regarding the recovery system is to switch on the timer circuit.

Unfortunately the location of the rocket after touchdown can be difficult, because the rocket can drift substantially when the parachute is deployed at apogee. A small pilot parachute or streamer can be used instead at apogee to guide the rocket fast down to avoid excessive drift. Then later the main parachute is deployed to achieve the necessary impact velocity. One drawback is the substantially more complex recovery system requiring two timers.

The recovery system divide the rocket with an internal separator in two parts respectively the engine and the payload section. The two parts are combined to the parachute with a line. The most important part is the payload section which carries the instruments to conduct the experiments. The line between the parachute and the engine is longest to achieve touchdown of the lesser important engine first. This arrangement reduce the impact velocity additional for the payload section.

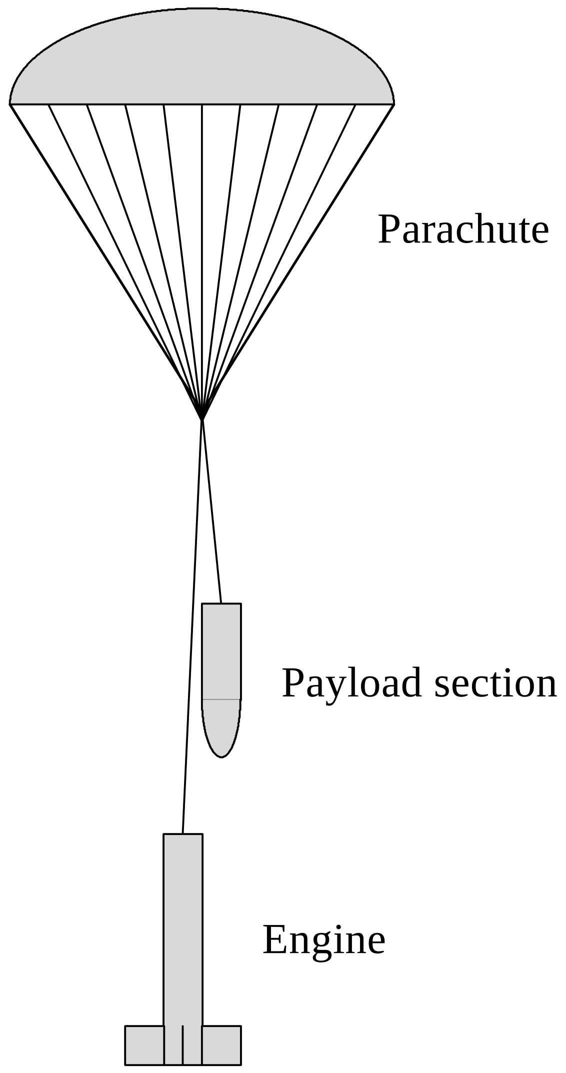

Fig. 2 Rocket travelling vertically downward in a parachute

Using different line lengths reduce the probability for impact between the engine and payload section when the rocket is divided. The lines are attached at the ends of the two sections. This assure mainly longitudinal stress at separation and touchdown on the sections.

2. Motion with air resistance equal to kv^2

The rocket in parachute (considered as a particle of mass m) having weight of magnitude mg is travelling vertically downward with the velocity v. The air resistance acting on the system has magnitude proportional to the square of the instantaneous velocity, and opposed to it.

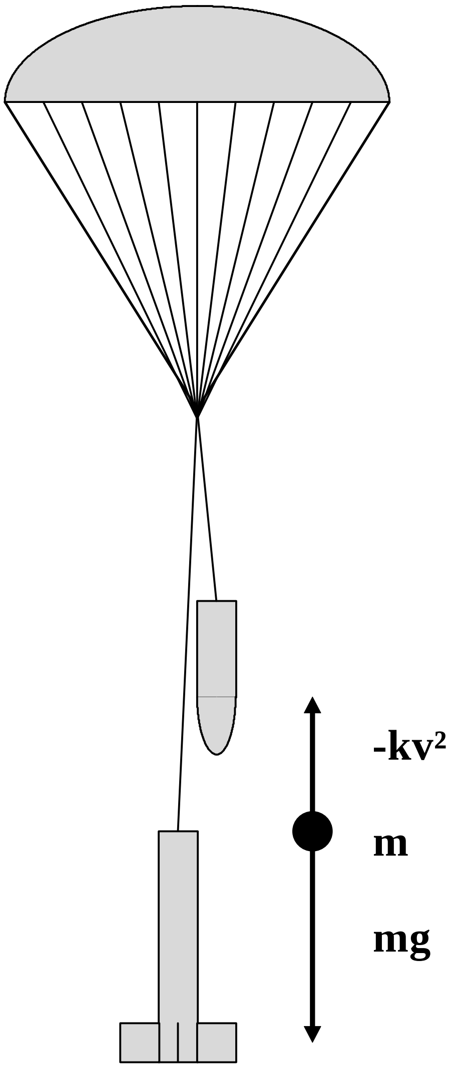

Fig. 3 Forces acting on a rocket travelling vertically downward in a parachute

Then by Newton's second law

$$ma = mg - kv^2 \Rightarrow$$

$$m\frac{dv}{dt} = mg - kv^2$$

which may be written

$$\frac{mdv}{mg - kv^2} = dt$$

Assuming g and k constant and that

$$v < \sqrt{\frac{mg}{k}}$$

we get by integrating

$$\frac{m}{k}\int_{v_0}^{v}\frac{dv}{\frac{mg}{k}-v^2} = \int_0^t dt \Rightarrow$$

$$\frac{m}{k}\left[\sqrt{\frac{k}{mg}}\tanh^{-1}\left(v\sqrt{\frac{k}{mg}}\right)\right]_{v_0}^{v} = t \Rightarrow$$

$$\sqrt{\frac{m}{kg}}\left(\tanh^{-1}\left(v\sqrt{\frac{k}{mg}}\right) - \tanh^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right) = t$$

The function $\tanh^{-1}(x)$ is the inverse hyperbolic tangent of x. The velocity at any subsequent instant is then

$$v = \sqrt{\frac{mg}{k}}\tanh\left(t\sqrt{\frac{kg}{m}} + \tanh^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right), v_0 < \sqrt{\frac{mg}{k}}$$

Similar we get

$$v = \sqrt{\frac{mg}{k}}\coth\left(t\sqrt{\frac{kg}{m}} + \coth^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right), v_0 > \sqrt{\frac{mg}{k}}$$

As t increases the hyperbolic tangent function approaches the limit 1 resulting in the limiting value for the velocity

$$v_{terminal} = \sqrt{\frac{mg}{k}}$$

This is called the terminal velocity, the limit to which the velocity of the parachute tends. It can also be obtained at once from the equation of motion. At the moment the air resistance balance gravity the acceleration a of the parachute is zero. Inserting in the equation of motion the same result for the terminal velocity is obtained.

The distance at any subsequent instant is

$$v = \frac{dy}{dt} \Rightarrow$$

$$\frac{dy}{dt} = \sqrt{\frac{mg}{k}} \tanh\left(t\sqrt{\frac{kg}{m}} + \tanh^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right) \Rightarrow$$

$$\int_0^y dy = \sqrt{\frac{mg}{k}} \int_0^t \tanh\left(t\sqrt{\frac{kg}{m}} + \tanh^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right) dt \Rightarrow$$

$$y = \sqrt{\frac{mg}{k}} \left[\frac{1}{\sqrt{\frac{kg}{m}}} \ln\left\{\cosh\left(t\sqrt{\frac{kg}{m}} + \tanh^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right)\right\} \right]_0^t \Rightarrow$$

$$y = \frac{m}{k}\left[\ln\left\{\frac{\cosh\left(t\sqrt{\frac{kg}{m}} + \tanh^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right)}{\cosh\left(\tanh^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right)}\right\}\right], v_0 < \sqrt{\frac{mg}{k}}$$

Similar we get

$$y = \frac{m}{k}\left[\ln\left\{\frac{\sinh\left(t\sqrt{\frac{kg}{m}} + \coth^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right)}{\sinh\left(\coth^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right)}\right\}\right], v_0 > \sqrt{\frac{mg}{k}}$$

Finally the acceleration at any subsequent instant is

$$a = \frac{dv}{dt} \Rightarrow$$

$$a = \sqrt{\frac{mg}{k}}\sqrt{\frac{kg}{m}}\left\{1 - \tanh^2\left(t\sqrt{\frac{kg}{m}} + \tanh^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right)\right\} \Rightarrow$$

$$a = g\left\{1 - \tanh^2\left(t\sqrt{\frac{kg}{m}} + \tanh^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right)\right\}, v_0 < \sqrt{\frac{mg}{k}}$$

Similar we get

$$a = g\left\{1 - \coth^2\left(t\sqrt{\frac{kg}{m}} + \coth^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right)\right\}, v_0 > \sqrt{\frac{mg}{k}}$$

3. Motion with air resistance equal to nv

At a relatively low velocity, the air resistance may be approximated by assuming that it is proportional to the instantaneous velocity, and opposed to it.

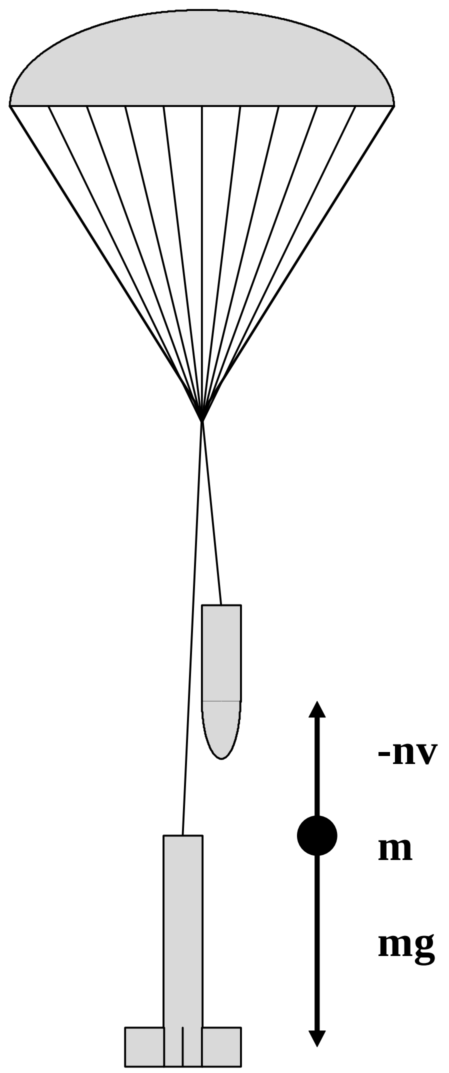

Fig. 4 Forces acting on a rocket travelling vertically downward in a parachute

The equation in this case is

$$ma = mg - nv \Rightarrow$$

$$m\frac{dv}{dt} = mg - nv$$

which may be written

$$\frac{mdv}{mg - nv} = dt$$

Assuming g and n constant, we get by integrating

$$m\int_{v_0}^{v}\frac{dv}{mg-nv}=\int_0^t dt \Rightarrow$$

$$\left[-\frac{m}{n}\ln(mg-nv)\right]_{v_0}^{v}=t \Rightarrow$$

$$\ln\left(\frac{mg-nv_0}{mg-nv}\right)=\frac{n}{m}t \Rightarrow$$

$$mg-nv_0=(mg-nv)\exp\left(\frac{n}{m}t\right)$$

The velocity at any subsequent instant is then

$$v=\frac{mg}{n}+\left(v_0-\frac{mg}{n}\right)\exp\left(-\frac{n}{m}t\right)$$

The terminal velocity is

$$v_{terminal}=\frac{mg}{n}$$

because as t increases the exponential function approaches the limit 0, as is likewise evident from the equation of motion.

The distance at any subsequent instant is

$$v=\frac{dy}{dt}\Rightarrow$$

$$\frac{dy}{dt}=\frac{mg}{n}+\left(v_0-\frac{mg}{n}\right)\exp\left(-\frac{n}{m}t\right)\Rightarrow$$

$$\int_0^y dy=\int_0^t\left(\frac{mg}{n}+\left(v_0-\frac{mg}{n}\right)\exp\left(-\frac{n}{m}t\right)\right)dt\Rightarrow$$

$$y=\left[\frac{mg}{n}t-\frac{m}{n}\left(v_0-\frac{mg}{n}\right)\exp\left(-\frac{n}{m}t\right)\right]_0^t\Rightarrow$$

$$y=\frac{mg}{n}t+\frac{m}{n}\left(v_0-\frac{mg}{n}\right)\left(1-\exp\left(-\frac{n}{m}t\right)\right)$$

Finally the acceleration at any subsequent instant is

$$a=\frac{dv}{dt}\Rightarrow$$

$$a = -\frac{n}{m}\left(v_0 - \frac{mg}{n}\right)\exp\left(-\frac{n}{m}t\right) \Rightarrow$$

$$a = \left(g - \frac{nv_0}{m}\right)\exp\left(-\frac{n}{m}t\right)$$

4. Determination of the coefficients k and n

The value k is equal to

$$k = ½ rAC_D$$

where

r : Density of the air.
A : Cross sectional area 5 a² of the cross-type parachute, where a is the length of a square.
C_D : Drag coefficient of the parachute (dimensionless).

The value n is equal to

$$n = hK$$

where

h : Coefficient of viscosity of the air.
K : Coefficient related to the shape of the parachute.

The coefficient η depends on the frictional force between different layers of the air moving with different velocities.

The coefficients C_D and K can be determined theoretical and experimental. The theoretical approach is complex because many parameters influence on the size of the coefficients as:

- Shape of the parachute.
- Porosity of the canopy material.
- Length of rigging lines.
- Interference from the rocket attached to the parachute.
- Interference from the rigging lines.

A practical approach is to derive formulas base on experimental data to obtain the coefficients.

The most common experimental technique to determine the coefficients is by the use of a wind tunnel. Practical limitations in the size of the wind tunnel and to avoid interference from the wind tunnel walls requires typically a scale model of the parachute to perform the measurements.

Another experimental approach is the vertical drop test from a building or tower where the actual parachute can be used. To avoid damage on the rocket itself a similar weight can be used as dummy. The coefficients can be determined by measuring the time taken by the parachute to fall a specified distance when released from rest. In the two cases we have the following relationship between distance and time when the initial velocity is zero:

$$y = \frac{m}{k}\ln\left\{\cosh\left(t\sqrt{\frac{kg}{m}}\right)\right\}$$

$$y = \frac{mg}{n}t - \frac{m^2 g}{n^2}\left\{1 - \exp\left(-\frac{n}{m}t\right)\right\}$$

Unfortunately the coefficients k and n can not be determined directly in the above equations by inserting the measured distance and time. In the latter equation the exponential function can be written as an infinite series as follows:

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

For small values of x this series converge very fast and therefore only the first four terms in the series are inserted in the equation, we get

$$y = \frac{mg}{n}t - \frac{m^2 g}{n^2}\left\{1 - (1 - \frac{n}{m}t + \frac{n^2}{2m^2}t^2 - \frac{n^3}{6m^3}t^3)\right\} \Rightarrow$$

$$y = \frac{1}{2}gt^2 - \frac{ng}{6m}t^3 \Rightarrow$$

$$n = \frac{3m}{t}\left(1 - \frac{2y}{gt^2}\right)$$

Unfortunately the above equations is more suited for objects with a fixed geometry during the whole fall. When the parachute is released from rest it takes some time before it get fully inflated causing inaccurate determination of coefficients. A more accurate method is to measure over a distance where the parachute travel with the terminal velocity, we have

$$y = v_{terminal}t$$

and in the two cases we get

$$k = mg\left(\frac{t}{y}\right)^2$$

$$n = mg\frac{t}{y}$$

5. Example

In this example we will use the formulas where the air resistance is proportional to the square of the instantaneous velocity. First we must determine the constant k for the parachute. The first problem you typically encounter using the vertical drop test is the demand for an extremely accurate timing device to achieve a reasonable accurate determination of k. Using a large drop distance y and a smaller mass m than the actual prolong the drop time avoiding an extremely accurate timing device.

The rocket have a burnout mass including the parachute equal to

$$m = 9.5\text{kg}$$

In the vertical drop test the rocket is replaced by a sandbag with a reduced weight to prolong the drop time. The mass of the sandbag including the parachute is equal to

$$m = 1.1\text{kg}$$

Finally you avoid to cushion the rocket at the end of its fall to avoid damage. A building is used with a drop distance of

$$y = 14.65\text{m}$$

Actually we dropped the parachute from a point 3m higher to assure that the parachute has achieved the terminal velocity at the beginning of the distance of interest. Several vertical drop tests resulted in an average drop time equal to

$$t = 3.6\text{s}$$

The constant k can now be found, we get

$$k = mg\left(\frac{t}{y}\right)^2 = 1.1\text{kg} \times 9.81\text{m/s}^2 \times \left(\frac{3.6\text{s}}{14.65\text{m}}\right)^2 = 0.65\text{kg/m}$$

The drag coefficient of the parachute is

$$C_D = \frac{2k}{rA} = \frac{2 \times 0.65\text{kg/m}}{1.225\text{kg/m}^3 \times 1.25\text{m}^2} = 0.85$$

where ρ is the density of the air and A is the cross sectional area of the cross parachute. The terminal velocity of the rocket is

$$v_{terminal} = \sqrt{\frac{mg}{k}} = \sqrt{\frac{9.5\text{kg} \times 9.81\text{m/s}^2}{0.65\text{kg/m}}} = 12\text{m/s}$$

Use a scaled parachute based on the one from the experiment to adjust the terminal velocity. This is possible because a scaled parachute do not influence on the size of the experimental found drag coefficient avoiding further experiments. A parachute twice the size used in the experiment reduce the terminal velocity by

$$\frac{1}{\sqrt{2}} \approx 0.7$$

We assume that the parachute move with the terminal velocity throughout the distance of interest, which is

$$v_{terminal} = \sqrt{\frac{mg}{k}} = \sqrt{\frac{1.1\text{kg} \times 9.81\text{m}/\text{s}^2}{0.65\text{kg}/\text{m}}} = 4.07\text{m}/\text{s}$$

To check this assumption we will find the velocity of the parachute when it enters the distance of interest. We first determine the time is takes to travel 3m

$$t = \sqrt{\frac{m}{kg}} \cosh^{-1}\left\{\exp\left(\frac{yk}{m}\right)\right\} \Rightarrow$$

$$t = \sqrt{\frac{1.1\text{kg}}{0.65\text{kg}/\text{m} \times 9.81\text{m}/\text{s}^2}} \times \cosh^{-1}\left\{\exp\left(\frac{3\text{m} \times 0.65\text{kg}/\text{m}}{1.1\text{kg}}\right)\right\} \Rightarrow$$

$$t = 1.02\text{s}$$

Remember the initial velocity is zero because we drop it and further we neglect the time it takes the parachute to fully deploy. Finally the velocity can be obtained after travelling 3m, we get

$$v = \sqrt{\frac{mg}{k}} \tanh\left(t\sqrt{\frac{kg}{m}}\right) \Rightarrow$$

$$v = \sqrt{\frac{1.1\text{kg} \times 9.81\text{m}/\text{s}^2}{0.65\text{kg}/\text{m}}} \tanh\left(1.02\text{s}\sqrt{\frac{0.65\text{kg}/\text{m} \times 9.81\text{m}/\text{s}^2}{1.1\text{kg}}}\right) \Rightarrow$$

$$v = 4.01\text{m}/\text{s}$$

Apparently the parachute have reached a velocity very close to the terminal velocity, so 3m is adequate to ensure this. Anyway we always force the parachute to be fully deployed before we release it, to create a scenario as close to the assumptions above.

Last we will determine the touchdown velocity of the payload. Due to different lengths of the lines respectively to the engine and payload, the payload section is 4m above ground at engine touchdown. This time the initial velocity is greater than the terminal velocity and the mass m is equal to

$$m = m_{rocket} - m_{engine} \Rightarrow$$

$$m = 9.5\text{kg} - 6.9\text{kg} = 2.6\text{kg}$$

As before we calculate first the time it takes to travel the distance, in this case 4m, we get

$$t=\sqrt{\frac{m}{kg}}\left[\sinh^{-1}\left\{\sinh\left(\coth^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right)\exp\left(\frac{yk}{m}\right)\right\}-\coth^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right]\Rightarrow$$

$$t=0.45\text{s}$$

Finally the velocity can be obtained after travelling 4m, we get

$$v=\sqrt{\frac{mg}{k}}\coth\left(t\sqrt{\frac{kg}{m}}+\coth^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)\right)\Rightarrow$$

$$v=7.3\text{m}/\text{s}$$

This simple arrangement reduce the touchdown velocity for the payload section from 12m/s to 7.3m/s. The terminal velocity of the payload is

$$v_{terminal}=\sqrt{\frac{mg}{k}}=\sqrt{\frac{2.6\text{kg}\times 9.81\text{m}/\text{s}^2}{0.65\text{kg}/\text{m}}}=6.3\text{m}/\text{s}$$

Finally I will not recommend to use the theory based on air resistance proportional to the instantaneous velocity in the above experiment. Especially the use of a reduced mass in the experiment to prolong the drop time result in a very inaccurate determination of the rockets terminal velocity. Inserting the values result in a terminal velocity of the rocket equal to

$$v_{terminal}=\frac{m_{rocket}\,y}{m_{\exp}t}\Rightarrow$$

$$v_{terminal}=\frac{9.5\text{kg}\times 14.65\text{m}}{1.1\text{kg}\times 3.6\text{s}}=39.5\text{m}/\text{s}$$

6. References

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