Trigonometric Identities

Reciprocal and Quotient Identities (§4.3 / §5.1)

$$\sin u = \frac{1}{\csc u} \qquad \cos u = \frac{1}{\sec u} \qquad \tan u = \frac{1}{\cot u} \qquad \tan u = \frac{\sin u}{\cos u}$$

$$\csc u = \frac{1}{\sin u} \qquad \sec u = \frac{1}{\cos u} \qquad \cot u = \frac{1}{\tan u} \qquad \cot u = \frac{\cos u}{\sin u}$$

Note

These identities are all true simply because of the way that the functions involved are defined.

Pythagorean Identities (§4.3 / §5.1)

$$\sin^2 u + \cos^2 u = 1 \qquad 1 + \tan^2 u = \sec^2 u \qquad 1 + \cot^2 u = \csc^2 u$$

$$\sin u = \pm\sqrt{1 - \cos^2 u} \qquad \tan u = \pm\sqrt{\sec^2 u - 1} \qquad \cot u = \pm\sqrt{\csc^2 u - 1}$$

$$\cos u = \pm\sqrt{1 - \sin^2 u} \qquad \sec u = \pm\sqrt{1 + \tan^2 u} \qquad \csc u = \pm\sqrt{1 + \cot^2 u}$$

(The choice of the sign in the identities involving a square root depends upon the quadrant in which the angle *u* lies, e.g. if *u* lies in the first or fourth quadrant, the cosine will be positive.)

Pattern

There are really only three identities here — those in the top row. The others can be derived by solving algebraically for the individual functions. In the first row, each of the three identities associates two functions and the number 1. The leftmost identity associates the sine and cosine functions; these are the only functions whose unit-circle definition (§4.2) involves no division. Each of the other two identities contains one function with the particle “tangent” and one function with the particle “secant” in its name. Both functions with the particle “co-” in their names belong to the same identity; both functions without the particle “co-” belong to the other identity. In both identities, the number 1 is on the same side as the (co)secant.

Alternately, it is possible to memorize only the first identity ($\sin^2 u + \cos^2 u = 1$) and to derive the other two identities from it by dividing both sides by dividing each term on both sides by $\cos^2 u$ (for the first identity) or $\sin^2 u$ (for the second identity), then simplifying.

Cofunction Identities (§4.3 / §5.1)

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

$$\cos\left(\frac{\pi}{2} - u\right) = \sin u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u \qquad \sec\left(\frac{\pi}{2} - u\right) = \csc u$$

Pattern

Cofunctions of complimentary angles are equal in value. (This is easily demonstrated on a right triangle by applying the definitions of the functions.)

Even/Odd Identities (§4.2 / §5.1)

$\sin(-u) = -\sin u$
$\csc(-u) = -\csc u$

$\cos(-u) = \cos u$
$\sec(-u) = \sec u$

$\tan(-u) = -\tan u$
$\cot(-u) = -\cot u$

Pattern

The cosine function and its reciprocal function, the secant function, are even; all others are odd.

Sum and Difference Formulas (§5.4)

$\sin(u + v) = \sin u \cos v + \cos u \sin v$

$\sin(u - v) = \sin u \cos v - \cos u \sin v$

$\cos(u + v) = \cos u \cos v - \sin u \sin v$

$\cos(u - v) = \cos u \cos v + \sin u \sin v$

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Sine-Sum Pattern

The right side of each identity has the form "sin *u* cos *v* ± cos *u* sin *v*." The operator in the middle of the right side is the same as the one on the left side of the identity.

Cosine-Sum Pattern

The right side of each identity has the form "cos *u* cos *v* ± sin *u* sin *v*." The operator in the middle of the right side is the opposite of the one on the left side of the identity.

Tangent-Sum Patterns

The numerator on the right side of each identity has the form "tan *u* ± tan *v*"; the operator is the same as on the left side of the identity. The denominator on the right side of each identity has the form "1 ± tan *u* tan *v*"; the operator here is the opposite of the operator inside the parentheses on the left side of the identity.

Double-Angle and Half-Angle Formulas (§5.5)

$\sin 2u = 2 \sin u \cos u$

$$\begin{aligned}\cos 2u &= \cos^2 u - \sin^2 u\\ &= 2\cos^2 u - 1\\ &= 1 - 2\sin^2 u\end{aligned}$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

$$\sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 - \cos u}{2}}$$

$$\cos\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan\left(\frac{u}{2}\right) = \frac{1 - \cos u}{\sin u}$$

$$\tan\left(\frac{u}{2}\right) = \frac{\sin u}{1 + \cos u}$$

Half-Angle Patterns

The numerators on the right side of sin $^u/_2$ and cos $^u/_2$ have the form 1 ± cos *u*, where the operator is addition for the cosine identity and subtraction for the sine identity. The second tangent identity can be formed from the first by taking the reciprocal and changing the operator in the (new) denominator from subtraction to addition.

As with the Pythagorean identities involving square roots, the choice of sign in the identities for sin $^u/_2$ and cos $^u/_2$ depends on the quadrant in which the angle $^u/_2$ lies.

Power-Reducing Formulas (§5.5)

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Pattern

The denominator in the sine and cosine formulas is 2; the numerator has the pattern "1 ± cos 2*u*". The operator is + for the cosine formula and - for the sine formula. The tangent formula is simply the sine formula divided by the cosine formula.

Product-to-Sum Formulas (§5.5)

$$\sin u \sin v = \frac{1}{2}\left[\cos (u - v) - \cos (u + v)\right]$$

$$\sin u \cos v = \frac{1}{2}\left[\sin (u + v) + \sin (u - v)\right]$$

$$\cos u \cos v = \frac{1}{2}\left[\cos (u - v) + \cos (u + v)\right]$$

$$\cos u \sin v = \frac{1}{2}\left[\sin (u + v) - \sin (u - v)\right]$$

Pattern

The right side of each of these identities has the form "½[*f*(*u* ± *v*) ± *f*(*u* ± *v*)]", where *f* is either the sine or the cosine function. When both functions on the left side of the identity are the same, *f* is the cosine function and the first operation is subtraction; when they are different, *f* is the sine function and the first operator is addition. The second (main) operator is addition when the second of the two functions on the left side of the identity is the cosine function and is subtraction when the second of the two functions on the left side is the sine function. The third operator on the right side of the identity is the opposite of the first operator: addition when the first operator is subtraction, subtraction when the first operator is addition.

Sum-to-Product Formulas (§5.5)

$$\sin x + \sin y = 2 \sin \left(\frac{x + y}{2}\right) \cos \left(\frac{x - y}{2}\right)$$

$$\cos x + \cos y = 2 \cos \left(\frac{x + y}{2}\right) \cos \left(\frac{x - y}{2}\right)$$

$$\sin x - \sin y = 2 \cos \left(\frac{x + y}{2}\right) \sin \left(\frac{x - y}{2}\right)$$

$$\cos x - \cos y = -2 \sin \left(\frac{x + y}{2}\right) \sin \left(\frac{x - y}{2}\right)$$

Pattern

The right side of each of these functions has the form "±2 *f*[½(*x* + *y*)] *g*[½(*x* - *y*)]", where *f* and *g* are the sine or cosine functions. The initial ± is a negative sign in the case of a difference of cosines, a positive sign otherwise. In the case of a sum or difference of cosines, both *f* and *g* are the same function; they are different functions in the case of a sum or difference of sines. When the operator on the left side of the identity is addition, the second function on the right side is the cosine function; when the operator on the left side is subtraction, the second function on the right side is the sine function.

Although this chart is keyed to the fifth edition of Larson & Hostetler's *Precalculus*, it should be useful for any student of college trigonometry.

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