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Philosophical remarks on three-valued logic

As it is well known, Jan Lukasiewicz invented his three-valued logic as a result of philosophical considerations concerning the problem of determinism. The fact that three-valued logic has its roots in philosophy, was very significant for Lukasiewicz himself. He wrote for example: „If there hadn't been some possibility that [...] the third value could be somehow intuitively interpreted, then the three-valued logic wouldn't probably have come into being". In this paper I would like to address not whether the introduction of the third logical value was sufficiently justified by some philosophical assumptions, but rather how it is possible to obtain the specific form of three-valued logic (so-called L_3 sentence calculus) from these assumptions alone. I think that this question reveals the greatest puzzle in the intuitive interpretation of the logical construal of Lukasiewicz. I will argue that the three-valued logic in the original Lukasiewicz's form contradicts some of the assumptions which allegedly has led to its invention.

First I shall try to present some basic notions and concepts of three-valued logic. Its crucial assumption is that besides the two ordinary logical values - i.e. truth and falsity - there is a third value, usually called "possibility". Sentences, which are "possible", refer to yet undetermined future states of affairs. On the other hand a sentence is true if it asserts a fact which is already determined, and a sentence is false, when the fact to which it refers is determined negatively - namely its negations is determined. But these assumptions are too weak for the construal of three-valued logic. We need rules which would connect the logical value of any compound sentence with the logical value of its components. Lukasiewicz gave these rules in the form of tables (or matrices) for the following connectives: negation (5) conjunction (ϖ), disjunction (ω) and implication (6). (Note that it is also possible to introduce new, non-classical modal connectives: „it is necessary that" (*L*) and „it is possible" (*M*), with the help of the following equations: Mp = 1 when p = 1 or 1/2 and Mp = 0 when p = 0; Lp = 1 when p = 1 and Lp = 0 when p = ½ or 0).

p	*q*	*p* ϖ *q*	*p* ω *q*	*p* 6 *q*
1	1	1	1	1
1	½	½	1	½
½	1	½	1	1
½	½	½	½	1
1	0	0	1	0
0	1	0	1	1
½	0	0	½	½
0	½	0	½	1
0	0	0	0	1

p	5*p*
1	0
0	1
½	½

Fig. 1. Truth-tables for logical connectives in L_3 calculus

Unfortunately Lukasiewicz did not present clear reasons for accepting such a semantical analysis of truth connectives in his three-valued logic. His declarations concerning this matter are highly enigmatic. Compare for example some quotations from his works: „The desired equations I obtained on the basis of detailed considerations, which were more or less plausible to me”[2]; „I was conducted by some intuitions and the willingness of saving some laws from the two-valued logic, such as: the law of identity, truth-conditions for conditionals, the rule of transposition”[3].

Nevertheless Lukasiewicz finally arrived at a non-classical logic, whose set of tautologies is not the same as the set of classical tautologies. More precisely, each tautology of the three-valued sentential calculus is a classical tautology, but not *vice versa.* The most spectacular examples of logical laws which are not valid in Lukasiewicz's calculus are of course: the principle of excluded middle and the principle of contradiction, but there are also other examples - as hypothetical sylogism in one of its forms: [(*p* 6 *q*) ϖ (*q* 6 *r*)] 6 (*p* 6 *r*). It is worth noting that with the one exception of the principle of contradiction, there are no arguments in Lukasiewicz's writings explaining why such-and-such law should not be valid in the three-valued logic.

Jerzy Slupecki tried to fill the gap between the assumption of the existence of the third value (together with other philosophical assumptions, for example the postulate of indeterminism) and the concrete form of calculus in his article.[4] He proposed a way of reconstructing Lukasiewicz’s three-valued calculus, by formalizing in a certain language his ontological assumptions. I am going to sketch the proposal of Slupecki. He assumed that on the set of all events Z (both present and future) it is possible to define the operations of sum, product and complement (negation), satisfying the axioms of a Boolean algebra. An additional relation in Z would be the causal relation, characterized formally by Slupecki with the help of five axioms. The last assumption is that all events split into three subsets: the set of positively determined, the set of negatively determined and the set of undetermined events.

The next step of Slupecki's construal should be introducing a special language with certain semantical features, enabling us to describe events of all three categories. Hence Slupecki assumes that every sentence from this language describes an event in Z. Then he defines truth-connectives in the following manner. Negation of a sentence *p* describes an event, which is a Boolean complement of the event described by *p,* the disjunction, „$p\ \omega\ q$", describes an event, which is a sum of events desribed by *p* and *q* separately; the conjunction, „$p\ \varpi\ q$", describes an event, which is a product of events described by p and *q.* Finally Slupecki introduces three logical values which can be assigned to each sentence - a sentence is true iff it describes a positively determined event; a sentence is false iff it describes a negatively determined event, and a sentence is possible (has the third logical value) iff it describes an event which is yet undetermined.

With these assumptions it is a question of simple reasoning to prove that truth-tables for above-mentioned connectives are identical with those from Lukasiewicz's calculus. Hence, if Slupecki's reinterpretation of philosophical assumptions of Lukasiewicz is correct, then it can be responsibly said that Lukasiewiczian calculus has an intuitive philosophical basis.

However, Slupecki's construal has some faults. First, it displays one formal defect. Although this defect is easy to eliminate, it is very characteristic that it has appeared in the context of three-valued logic. It turns out namely that the initially accepted assumption that the structure Z is a Boolean algebra is too strong, and in conjunction with other assumptions leads to a contradiction. It is so because in a Boolean algebra there is always a unique maximal element, usually denoted by „1": $1 = f + f'$, where „+" denotes the Boolean sum, and ' - the Boolean complement. Now let us take an event f_1 which has been positively determined and an event f_2 which is undetermined. From the equation $f_1 + f_1' = 1$ and the condition characterizing the causal relation it follows that the event denoted by "1" will be also positively determined (the condition in question states that the sum of events is positively determined if and only if at least one of these events is positively determined). But because in a Boolean algebra $1 = f_2 + f_2'$ also holds, then on the basis of the same asumption we get that either f_2 or f_2' is positively determined, which contradicts the initial assumption. This difficulty can be easily overcome by weakening the axioms of a Boolean algebra to the axioms of a de Morgan lattice.[5] But later we will see that there are no good philosophical reasons for which the operation $f + f'$ should give different results, depending on what kind of event f is.

The second, and more serious problem with Slupecki's proposal is that it does not include implication. As is known, in the three-valued logic, the implication connective is not definable in terms of conjunction and negation, or disjunction and negation. It is an interesting fact that Slupecki did not succeed in finding an operation on events which would be a counterpart to implication - therefore his construction is not complete. And I think it is not a pure coincidence - because implication in three-valued logic causes more conceptual difficulties.

Now let us move to the critical analysis of the three-valued calculus of Lukasiewicz. I will present two arguments against its intuitive interpretation. The first was raised during the philosophical congress in Zurich, where Lukasiewicz presented his discovery. The objection consists of a very simple observation: when a sentence *p* concerning some future event has today a value of 1/2, then according to the appropriate truth-tables the sentence „*p* ω 5*p*" also has the third value, which means that it is not yet determined whether the state of affairs denoted by the latter formula will really take place (the same goes with the formula „*p* ϖ 5*p").* But this goes against the very basic observation that independently of the value of *p,* today it is already determined that the first sentence is true and the second - false. Using famous Aristotelian example we can say that perhaps the sentence: "Tomorrow a sea battle will take place" is neither true nor false, yet the sentence: "Either tomorrow will be a sea battle or tomorow will not be a sea battle" is certainly true. In other words, there is no explanation for rejecting the principle of excluded middle and the principle of contradiction on the basis of the assumption of indeterminism.

Ludwik Borkowski treats this argument as conclusive.[6] In his article he complements this objection by the observation that in Lukasiewicz's calculus the following modal formula is valid: (*Mp* ϖ *Mq*) 6 *M(p* ϖ *q),* which „says” that if two sentences are possible, then their conjunction is also possible. But this is obviously untrue: if one of them is a negation of the second, then although both can be possible, their conjunction is always impossible. Borkowski proposes the following correction of the Lukasiewiczian calculus. Two facts must be reconciled: the fact that the conjunctions and disjunctions of two possible sentences "usually" are possible, with the fact that the conjunction of one possible sentence with its negation is nevertheless false, and their disjunction is true. Borkowski claims that this reconciliation is possible only on the ground of a four-valued logic. Therefore he introduces two intermediate values between truth and false (we can symbolize them by 2 and 3), and accepts the following theorems: (1) if a sentence has the value 2, then its negation has the value 3, (2) if a sentence has the value 3, then its negation has the value 2, (3) the disjunction and conjunction of two sentences both of value 2 have the value 2, and the disjunction and conjunction of sentences having value 3 have the value 3, (4) the disjunction of sentences with different intermediate values (2 and 3) is true, and their conjunction false. It is easy to check that in this system the above-mentioned difficulty disappears.

But this solution can hardly be accepted as satisfactory. First of all it is *ad hoc*, because the author does not explain the difference between the two intermediate logical values. Moreover, the new system is open to almost the same criticism as Lukasiewicz’s original calculus. Let us for example consider two semantically independent sentences *p* and *q*, both being "possible", which means that both of them refer to some future unrelated possible events. On the basis of Borkowski's assumptions it can be shown that from two formulas: „*p* ω *q”* and „*p* ω 5*q”,* one must be true. It is so, because either *p* and *q* have the same value, or they do not. If it is the first case - then *p* and 5*q* have different values and therefore „*p* ω 5*q”* is true If the second, then „*p* ω *q”* must be true. Hence the case is proved. But this conclusion is not acceptable either, for both sentences are independent from each other and their disjunctions should also only be possible.

So this proposal does not defend Lukasiewicz's logic against the first objection.

Nevertheless there is also a second one, which we may call „the paradox of implication". According to the intuitive interpretations of truth values in the three-valued logic, possible sentences - referring to the undetermined future -in a future moment will be „realized", and thereby be determined as true or false. So in the three-valued logic sentences can change their truth values in time, but only from 1/2 to 1 or 0. Suprisingly enough, it is a very simple task to construct a sentence which will change its value from 1 to 0. Let us take two sentences *p* and *q* undetermined at the moment *t,* and let us assume that after some time the first becomes true and the second false. Now consider the implication „*p* 6 *q*" According to the truth-table, its value at the moment *t* is 1, but then it comes down to 0. But how does one reconcile this fact with the intuitive interpretation of the truth-values? It turns out that an event can change from being determined positively to being determined negatively.

I think that now we are in a position to formulate the following two theses: (1) three-valued calculus cannot be completely reconstructed from intuitive philosophical assumptions alone; (2) the concrete form of Lukasiewicz's calculus is not compatible with its philosophical interpretation. Does it necessary mean that it is impossible to formalize, logically, Lukasiewicz's intuitions in three-valued logic? I do not think so. But in order to achieve this, we must create a new calculus which would lack some properties of ordinary sentential calculus. Now I would like to sketch the ideas of this new calculus. Let us accept that $p_1, p_2, ..., p_n$ are sentences which refer to some mutually independent future events. We will accept that every sentence has one of the three values as above. In order to calculate the truth value of the complex sentence, we must follow a certain procedure. Let $\alpha(p_1, p_2, \ldots, p_k)$ be any complex sentence. If the sentences p_1, p_2, …, p_k have classical values, the value of α would be the same as in classical logic (calculated with the help of usual two-valued truth-tables). But if some of the atomic sentences are possible (the truth-value 1/2) then we should substitute for each of them 1 or 0 and calculate the truth-value for each case. If the outcome is always 1, then the final result is 1, if 0 then 0 - but if in some cases the result is 1 and in some 0, then the final value is 1/2.

Let us illustrate this with some simple examples. (1) *p* ω 5*p* = 1 if *p* = 1/2, (2) *p* ϖ 5*p* = 0 if *p* = 1/2, (3) *p* ω *q* = 1/2 and *p* ϖ *q* = 1/2 if *p, q* = 1/2, (4) *p* 6 *q* = 1/2 if *p,q* = 1/2. (5) *p* 6 *p* = 1, if

p = 1/2. These examples show that all conceptual difficulties of ordinary three-valued logic disappear. It is easy to show that each classical tautology remains a logically true sentence in our language, and that there are no other tautologies than the classical ones. However, this calculus is not extensional in this sense that the truth value of a complex sentence is not in general preserved under substitution.[7]

I will finish with the remark that is possible to augment this calculus by the modal notions „it is necessary" and „it is possible". This requires the modification of the above presented rule of calculating logical values of complex formulas. Roughly speaking, we should first calculate the logical value of each subformula with the range of some modal operator, and then apply the usual Lukasiewicz's truth-tables. In this way it is possible to prove that most of the usual logical laws including modalities are valid within this system. But I will not go into these, rather technical, details.

Notes

1. J. Lukasiewicz, "Geneza logiki tr6jwartosciowej" ("A genesis of three-valued logic"], *Filozofia Nauki [Philosophy of Science],* 3-4(7-8) 1994, p. 234

2. J. Lukasiewicz, „Uwagi filozoficzne wielowartosciowych systemach rachunku zdan" [„Philosophical Remarks on Many-Valued Systems of Propositional Logic"), in: Z *zagadnien logiki i filozofii [On logical and philosophical questions],* PWN, Warszawa 1961, pp. 153-154.

3. J. Lukasiewicz, „Geneza logiki tr6jwartosciowej", *op. cit.,* p. 239.

4. Cf. J. Slupecki, „Proba intuicyjnej interpretacji logiki tr6jwarto5ciowej Lukasiewicza" ["An attempt of an intuitive interpretation of Lukasiewicz's three-valued logic') in: *Rozprawy logiczne [Logical essays),* PWN, Warszawa 1964, pp. 185-191.

5. This is done in details in: M. Nowak, "0 mozliwosci interpretowania trojwartosciowej logiki

Lukasiewicza metoda J. Slupeckiego" ["On the possibility of interpretation of Lukasiewicz's three-valued logic employing J. Slupecki's method"], *Acta Universitatis Lodziensis. Folia Philosophica* 5, 1988, pp. 3-13,

6. Cf. L. Borkowski, "W sprawie intuicyjnej interpretacji logiki tr6jwartosciowej Lukasiewicza" ["On the question of an intuitive interpretation of three-valued logic of Lukasiewicz"] in: *Studia logiczne [Logical Studies],* TN KUL, Lublin 1990, p. 428.

8. Short after finishing the first version of this paper, I found out that in fact the logical calculus of a very similar type has already been invented. It is namely van Fraassen’s „presuppositional language”, which has the same semantical rules as the calculus described above, assigning truth-values to complex sentences with the help of the so-called supervaluation function. See: S. Haack, *Deviant Logic, Fuzzy Logic,* The University of Chicago Press, Chicago & London 1996, p. 85. But, as far as I know, van Fraassen’s calculus do not embrace modal operators.